Part I. Definitions/Theorems.
1. State the structure theorem for finitely generated modules over a PID.
2. What does it mean to say that a functor $F$ is left adjoint to a functor $G$?
   State the theorem about adjointness of $\otimes$ and hom.
3. State Noether’s normalization lemma, making sure you include the definition of an integral ring extension along the way.

Part II. True/False. Justify your answers briefly.
1. All groups of order 18 are nilpotent.
2. If $R$ is a ring with no non-trivial two-sided ideals, then it is a division ring.
3. The Jacobson radical of a ring $R$ is the largest nilpotent ideal of $R$.
4. If $G$ is a finite group, $P$ is a Sylow $p$-subgroup, and $P \leq N \leq G$, then $P \leq G$.
5. $\mathbb{C}[x, x^{-1}]$ is a projective $\mathbb{C}[x]$-module.
6. If $F$ is an algebraically closed field and $I, J$ are ideals in $F[x_1, \ldots, x_n]$, then $V(I) \cup V(J) = V(IJ)$.

Part III. Longer problems. Attempt any FOUR of the following five questions.
1. Let $G_1 = S_3 \times C_2$, let $G_2 = C_6 \times C_2$ where the semidirect product is defined so that the non-trivial element of $C_2$ acts on $C_6$ as the map $x \mapsto x^{-1}$, and let $G_3$ be the group with generators and relations $(s, t | s^2 = t^2 = 1, ststs = tsts)$. Prove that $G_1 \cong G_2 \cong G_3$.
2. Explaining your reasoning, compute the Jordan normal form of the matrix
   \[
   \begin{pmatrix}
   0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
   0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
   0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 
   \end{pmatrix}
   \]
3. Let $V$ be a finite dimensional vector space over a field $F$. Let $f, g : V \to V$ be linear maps that generate a proper subalgebra of the endomorphism algebra $\text{End}_F(V)$. Prove that there is a non-zero proper subspace $U < V$ left invariant by both $f$ and $g$.
4. Let $V$ be the vector space $\mathbb{R}^n$ equipped with the usual inner product $(.,.)$. Let $G$ be a (not necessarily finite) subgroup of the orthogonal group $O(V) = \{g \in GL(V) | (gv, gw) = (v, w) \text{ for all } v, w \in V\}$. Prove that $V$ is a semisimple $\mathbb{R}G$-module.
5. Let $M$ be an $R$-module of finite length $n$ and $f \in \text{End}_R(M)$ be an endomorphism. Prove that $M = \text{im} f^n \oplus \ker f^n$. 

1. State the structure theorem for f.g. modules over a PID $R$.

Let $M$ be f.g. $R$-module. Then, $M$ can be decomposed as

$$M = M_1 \oplus \cdots \oplus M_n$$

Each $M_i \cong R/(d_i)$ for non-unit $d_i \in R$ s.t.

$$d_1 | d_2 | \cdots | d_n$$

The number $n$ and $d_1, \ldots, d_n$ are unique (up to associates).

2. What does "$F$ is left adjoint to $G" mean? State theorem about adjoints.

Say $F: A \to B$, $G: B \to A$.

Means there's a natural isomorphism

$$\text{Hom}_A(M, GN) \cong \text{Hom}_B( FM, N)$$

i.e., the bifunctor $\text{Hom}_A(?, GN)$ and $\text{Hom}_B(FM, ?)$ are $\cong$.

If $R M S \otimes _R N$ and $R P$ for any $R, S$, then there's a natural isomorphism

$$\text{Hom}_R(R M S \otimes _R N, R P) \cong \text{Hom}_S(S N, \text{Hom}_R(R M S, R P))$$

i.e., $R M S \otimes _R N$ is left adjoint to $\text{Hom}_R(R M S, ?)$.

3. Let $R$ be an integral domain that's also an $F$-algebra, $F$ a field.

Then, $F$ algebra-independent elements $k_{1, \ldots, t} \in R$ s.t. $R$ is an integral extension of $F[k_{1, \ldots, t}]$.

(\text{ means: every elt } \in R

is not a nonic poly with

coefficients in } F[k_{1, \ldots, t}]$.
1. All groups order 18 are nilpotent.
   1. Take non-abelian semidirect product \((C_3 \times C_3) \rtimes C_2\) swapping \(C_3\) factors.
   Not nilpotent, since if it were it would be product of Sylow \(p\)'s, \(C_3 \times C_3 \times C_2\) hence abelian!

2. If \(R\) has no non-trivial 2-sided ideals, it is a division ring.
   2. Take \(M_2(F)\), \(F\) a field. It simple, not a division ring.

3. For any \(R\), \(J(R)\) is largest nilpotent ideal.
   3. (true for Artinian rings). Take \(2_{(p)} = 2 \) localised at \(\mathfrak{p} = \{ \frac{a}{b} | \mathfrak{p} b \} \).
      In a local ring unit 1 max ideal \(\{ \frac{a}{b} | \mathfrak{p} a, \mathfrak{p} b \} \). This must be the Jacobson radical stupidly.
      Is not a nilpotent ideal.

4. \(G\) finite, \(P \trianglelefteq G\), \(P\) Sylow \(p\)-subgroup \(\implies G \Rightarrow P \trianglelefteq G\).
   4. As \(P \trianglelefteq G\), it unique Sylow \(p\)-subgroup \(\equiv N\), so a characteristic subgroup.
      For \(x \in G\), \(x^p x^{-1} \leq N\) must normal be \(P\) again. So \(P \trianglelefteq G\).

5. \(\mathbb{Q}(x, x^{-1})\) is a projective \(\mathbb{Q}(x)\)-module.
   5. Can show \(\mathbb{Q}(x) \rightarrow \mathbb{Q}(x, x^{-1})\).
      \((f_0(x), f_1(x), \ldots) \rightarrow f_0(x) + x f_1(x) + x^2 f_2(x) + \cdots\).
      If projective, this would split, so have \(B: \mathbb{Q}(x, x^{-1}) \rightarrow \bigoplus_{\mathbb{Q}(x)}\) splitting.
      Say \(B(1) = (f_0(x), f_1(x), \ldots)\)
      \(\Rightarrow \) each \(f_c(x)\) has constant ten zero (or something).
      \(\Rightarrow \) each \(f_c(x)\) has exactly ten zero
      \(\Rightarrow \) each \(f_c(x) = 0\) \(\Rightarrow B(1) = 0\) \# not a splitting.

6. \(V(IJ) = \bigvee(IJ)\)
   6. Take \((x), (y) \in \mathbb{Q}(x, y)\) \(IJK = (xy)\) so \(V(IJK) = \) union \((x+y)\) axis.
      But \(V(IJK) = \) only when \(x, y\) vanishes = origin only.
$G_1: S_3 \times C_2, \quad G_2: C_6 \times C_2, \quad G_3: \langle s, t \mid s^2 = t^2 = 1, \text{stssts}=\text{tststs} \rangle$

Prove $G_1 \cong G_2 \cong G_3$.

I'll show they all $\cong D_6$, dihedral group of order 12.

$D_6 = \langle x \rangle \times \langle x, y \rangle \cong C_2 \times S_3$

$G_2: \langle x \rangle$ is normal, index 2, $\langle y \rangle$ is complement.

$D_6 = \langle x \rangle \times \langle y \rangle \cong C_2 \times C_2$.

Let $z = yxy = \text{next-door axis}$.

$y^2 = z^2 = 1, \quad yz = z^{-1}$.

$\langle s, t \mid s^2 = t^2 = 1, \text{stssts}=\text{tststs} \rangle \rightarrow D_6$

$s \rightarrow x$

$t \rightarrow y$

Remain to show this group has order at most 12.

Everything reduces any $s^2t^2 = 1$ to $s \text{stssts} \rightarrow \ldots$ or $t \text{ststs} \rightarrow$

Using $tst^6 = 1$ if have $\text{ststs}$ as subword, $s \neq \text{stststs}$ shorter.

Similarly for $\text{tststs}$.

Reduce to $1, s, st, sts, stt, stts, tsts, tstst, tstst, tststs, tsstst, tsststs$. (12 elements)
Compute JNF $\gamma$

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \end{bmatrix}$$

Note $A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \end{bmatrix}$

$A^3 = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \end{bmatrix}$

$\text{dim ker } A = 4$ $\text{dim ker } A^2 = 6$ $\text{dim ker } A^3 = 8$ $\text{dim ker } A^4 = 9$ $A^5 = 0$

$\therefore$ Jordan $\chi$-algebra, $2\gamma$ size 1, $\alpha$ size 5 $3\gamma$ size 4
$\gamma$ size 5, $\lambda$ size 3, $2\gamma$ size 1
2. $\text{fige } \text{End}_F(V)$ s.t. $\langle \text{fig} \rangle$ is proper subalgebra of $\text{End}_F(V)$.

Prove there's a non-zero $U \subseteq V$ left invariant by both $\text{fig}$ and $\text{fig}$.

Suppose not, so $V$ is semisimple.
Let $A = \langle \text{fig} \rangle$ -module.

Brauer's theorem $\Rightarrow A \cong \text{End}_F(V)$.

$A = \text{End}_F(V)$ by dimension.

Proof of this: Say $\dim V = n$.

From Wedderburn, as a left module, $\text{End}_F(V) \cong \underbrace{V \oplus \cdots \oplus V}_{n \text{ times}}$.

As a $A$-module, as $V$ is simple.

A $A$ is simple $A$-module.

$V$ is only constituent, so only used $A$-module.

Wedderburn $\Rightarrow A \cong \text{End}_F(V)$. 

5. Prove Fitting's lemma. \( f : M \rightarrow M \) of length \( n \).

\[ \text{Img} \supseteq \text{Img}^2 \supseteq \ldots \supseteq \text{Img}^n \]

As soon as equal once, equal forever, so suit \( M \) has length \( n \), \( \text{Img}^n = \text{Img}^{n+1} = \ldots \).

\[ \therefore \text{Img}^n = \text{Img}^{2n} \text{. Take } x \in M. \text{ } f^n(x) = f^{2n}(y) \text{ some } y. \]

Hence \( x = f^n(y) - (x - f^n(y)) \).

\[ M = \text{Img}^n + \ker f^n \]

Similarly, \( \ker f \subseteq \ker f^2 \subseteq \ldots \) so \( \ker f^n = \ker f^{n+1} = \ldots = \ker f^{2n} \).

Take \( x \in \text{Img}^n \cap \ker f^n. x = f^n(y) \text{ some } y, \text{ } f^n(x) = f^{2n}(y) = 0 \)

\[ \therefore y \in \ker f^{2n} = \ker f^n \]

\[ \therefore x = f^n(y) = 0 \text{ } \therefore \text{Img}^n \cap \ker f^n = \{ 0 \} \]