Algebra Qualifying Exam

Winter 2005

Conventions: Throughout this examination, assume all rings have identities and all modules are unitary.

Part I Theorems. Carefully state each of the following;

1. The structure theorem for finitely generated modules over a PID.
2. The Jacobson density theorem.
3. The fundamental theorem of Galois theory.

Part II True–False. Determine whether each statement is true or false. Support your decision by a brief explanation or a counterexample.

1. There are three pairwise nonisomorphic groups of order 121.
2. Over a PID every finitely generated projective module is free.
3. Over $\mathbb{Z}$ the image of an injective module under a homomorphism is again injective.
4. Let $\mathbb{Z}_2$ be a $\mathbb{Z}_6$-module via the projection $\mathbb{Z}_6 \to \mathbb{Z}_6/2\mathbb{Z}_6 = \mathbb{Z}_2$. Then $\mathrm{Hom}_{\mathbb{Z}_6}(\mathbb{Z}_2, \mathbb{Z}_6)$ has 3 elements.
5. A ring R is simple if and only if it is a matrix ring over a division ring.
6. If u and v are algebraic over $\mathbb{Q}$, with $u \notin \mathbb{Q}(v)$ and $v \notin \mathbb{Q}(u)$, of degrees m and n respectively, then $[\mathbb{Q}(u, v) : \mathbb{Q}] = mn$.
7. The image of the Jacobson radical of a ring under any ring homomorphism is the Jacobson radical of the image.

Part III Problems. Give complete solutions for each of the following.

1. Compute the localizations of the ring $\mathbb{Z}_{12}$ at its maximal ideals.
2. Classify all groups of order 15.
3. Let I be an ideal of a ring R lying in the Jacobson radical of R and let M be a finitely generated R-module. Then $IM = M$ implies $M = 0$.
4. Let I be the ideal of $\mathbb{R}[x, y, z]$ consisting of all the polynomials p such that $p(x, 0, 0) = \partial p/\partial y(x, 0, 0) = \partial p/\partial z(x, 0, 0) = 0$ for every x. Find a primary decomposition of I and all the associated primes of I.

Algebra Qualifying Exam - Solutions

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Conventions: Throughout this examination, assume all rings have identities and all modules are unitary.

Part II True–False. Determine whether each statement is true or false. Support your decision by a brief explanation or a counterexample.

1. There are three pairwise nonisomorphic groups of order 121.
False: Any group of order p^2 is Abelian (by the class equation) and hence there are only two groups of order $121 = 11^2$, $\mathbb{Z}_{121}$ and $\mathbb{Z}_{11} \times \mathbb{Z}_{11}$ (be the Fundamental Theorem of finitely generated Abelian groups, or modules over a PID).
2. Over a PID every finitely generated projective module is free.
True: FInitely generated torsion free modules are free, and projectives are, at the very least, submodules of free and hence torsion free modules.
3. Over $\mathbb{Z}$ the image of an injective module under a homomorphism is again injective.
True: Over $\mathbb{Z}$, injectivity is the same as divisibility, which is clearly preserved by homomorphic images.
4. Let $\mathbb{Z}_2$ be a $\mathbb{Z}_6$-module via the projection $\mathbb{Z}_6 \to \mathbb{Z}_6/2\mathbb{Z}_6 = \mathbb{Z}_2$. Then $\text{Hom}_{\mathbb{Z}_6}(\mathbb{Z}_2, \mathbb{Z}_6)$ has 3 elements.
False: There are only two elements of $\mathbb{Z}_6$ annihilated by 2, i.e. $\bar{0}$ and $\bar{3}$, and thus only two possible homomorphisms.
5. A ring R is simple if and only if it is a matrix ring over a division ring.
False: The Weyl algebra $A_1(\mathbb{C}) := \mathbb{C}[t, \frac{d}{dt}]$ is an example of a simple ring that is not a matrix ring over a division ring.
6. If u and v are algebraic over $\mathbb{Q}$, with $u \notin \mathbb{Q}(v)$ and $v \notin \mathbb{Q}(u)$, of degrees m and n respectively, then $[\mathbb{Q}(u,v) : \mathbb{Q}] = mn$.
False: Let $u = \sqrt{2} + \sqrt{3}$ and $v = \sqrt[4]{6}$, which each have degree 4 (since $\mathbb{Q}(u)$ has $\mathbb{Q}$-basis $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$. Thus $[\mathbb{Q}(u,v), \mathbb{Q}] = [\mathbb{Q}(u,v) : \mathbb{Q}(u)] \cdot [\mathbb{Q}(u) : \mathbb{Q}] = 2 \cdot 4 = 8 \neq 16$.
7. The image of the Jacobson radical of a ring under any ring homomorphism is the Jacobson radical of the image.
False: $J(\mathbb{Z}) = 0$, but $J(\mathbb{Z}/4\mathbb{Z}) = 2\mathbb{Z}/4\mathbb{Z} \neq 0$.

Part III Problems. Give complete solutions for each of the following.

1. Compute the localizations of the ring $\mathbb{Z}_{12}$ at its maximal ideals.
$R := \mathbb{Z}_{12}$ has exactly two primes: $P := 2R$ and $Q = 3R$ (since these are the only (necesarrily cyclic) ideals which pull back to primes in $\mathbb{Z}$. Both primes are maximal. To compute R_P we must invert elements not in P, thereby annhilating anything that is $(R \setminus P)$-torsion, namely, $4R$ (since $\bar{3} \notin P$). But $R/4R$ is already local. Thus $R_P = (R/4R)_P = R/4R = \mathbb{Z}_4$. Similarly, to compute R_Q we must first annihilate $3R$, and obtain $R_Q = (R/Q)_Q = R/Q = \mathbb{Z}_3$.
2. Classify all groups of order 15.
By the Sylow theorem, since 5 mod 3 is 2, any group of order 15 must have a normal Sylow 5-subgroup, isomorphic to $\mathbb{Z}_5$ and a normal Sylow 3-subgroup isomorphic to $\mathbb{Z}_3$. Since both are normal, G is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_5$.

3. Let I be an ideal of a ring R lying in the Jacobson radical of R and let M be a finitely generated R-module. Then $IM = M$ implies $M = 0$.

Suppose M is cyclic, say $M = Rx$. then $IM = Ix = M$. In particular $x = rx$ for some r in $J(R)$. But $r \in J(R)$ if and only if $1 - sr$ is invertible for all $s \in R$. So there is a y in R with $y(1-r) = 1$. But then $x = y(1-r)x = 0$, so M is 0.

Proceed by induction on the number of generators n of M. If $M = Rx_1 + \ldots Rx_n$, then let $N = Rx_1 \cdots Rx_{n-1}$ and consider M/N. $I(M/N) = (IM + N)/N = M/N$. But M/N is cyclic, so $M/N = 0$. This means that M actually had $n-1$ generators, so by induction $M = 0$.

4. Let I be the ideal of $\mathbb{R}[x, y, z]$ consisting of all the polynomials p such that $p(x, 0, 0) = \partial p/\partial y(x, 0, 0) = \partial p/\partial z(x, 0, 0) = 0$ for every x. Find a primary decomposition of I and all the associated primes of I.

The ideal is clearly $I = (y^2, yz, z^2)$. We note that $I = J^2$, with $J = (y, z)$. An associated prime of I is any prime annihilator of a nonzero element of R/I. J is the annihilator of $y + I$ (pf: if $py \in I$, then $\frac{\partial p}{\partial y}y + p$ must vanish on $(x, 0, 0)$, so p vanishes and is in J) and J is prime, so J is an associated prime. Suppose K is the annihilator of some other element $q + I$, we may assume $q = a + bx + cy$. If $a \neq 0$ then $K = I$. If $a = 0$, the J anihilates q and so $J = K$ (pf: if K is strictly bigger than J then it contains a pure polynomial in x, which can't annihilate anything in R/I) . Thus J is the only associated prime, from which it follows that I is already primary and is its own primary decomposition.