ALGEBRA QUALIFYING EXAM, WINTER 200~~3~~4

Conventions: All rings have identity, and all modules are unital.

Part I. Carefully state each of the following (be certain that any *special* terminology or notation is explained):

1. The fundamental theorem on finitely generated abelian groups.
2. The fundamental theorem of Galois Theory.
3. What is equivalence of categories?

Part II. For each of the following determine whether it is true or false. If true, give a brief explanation. If false, provide a counterexample:

1. $Q_8 \cong C_4 \rtimes C_2$.
2. Classify the groups of order 175.
3. Let $K/\mathbb{F}_q$ be a finite extension, and L, M be two intermediate subfields. Then either $L \subseteq M$ or $M \subseteq L$.
4. $\mathbb{Z}_{35} \otimes_{\mathbb{Z}} \mathbb{Z}_5 \cong \mathbb{Z}_7$
5. Let F be algebraically closed. Any increasing sequence of irreducible algebraic sets in F^n stabilizes.
6. If R is a commutative artinian ring, then $R[x]$ is noetherian.
7. If I is an ideal of a commutative ring R such that $\sqrt{I}$ is a maximal ideal in R then I is primary.

Part III. Give complete solutions for each of the following problems.

1. Let G be a finite group G, and $g_1, \ldots, g_k$ be representatives of the conjugacy classe of G. Then $G = \langle g_1, \ldots, g_k \rangle$.
2. Let K be a splitting field for $x^4 - 3$ over $\mathbb{Q}(i)$ (where $i = \sqrt{-1} \in \mathbb{C}$). Find the Galois group of K over $\mathbb{Q}(i)$.
3. Let V be a finite dimensional vector space over a field K.
   (a) Exhibit a natural isomorpism of vector spaces
   $$\beta : V^* \otimes_K V \to \mathrm{Hom}_K(V, V);$$
   (b) Exhibit a natural linear map $\theta : V^* \otimes_K V \to K$.
   (c) If $F \in \mathrm{Hom}_K(V, V)$, prove that $\mathrm{Trace}(F) = \theta(\beta^{-1}(F))$.
4. Let R be a domain and F be its field of fractions. Prove that F is an injective R-module.
5. Say all you can about a finite simple ring.