Algebra Qualifying Exam, Winter 2001

Assume all rings have identity elements.

Section 1: State the theorems below, defining relevant terms.

i. The Jacobson density theorem.
ii. The theorem describing a matrix in terms of its rational canonical form.
iii. The fundamental theorem of Galois theory.

Section 2: True/False. If FALSE, provide a counterexample, if TRUE, give a brief justification.

a. If R is a commutative ring, any submodule of a free module is itself free.
b. Every group has a non-trivial center.
c. A finite linear transformation L on a vector space V of characteristic 0 is nilpotent if and only if the trace of L is 0.
d. Any decreasing sequence of varieties of K^n, $V_1 \supseteq V_2 \supseteq \ldots$ stabilizes.
e. R is simple if and only if $R \cong \mathrm{Mat}_n(D)$ for D a division algebra.
f. If R is commutative, and every submodule of a free module is free, then R is a P.I.D.

Section 2: Give complete proofs for 4 problems from the following.

(1) Prove that if $0 \to A \to B \to C \to 0$ is a short exact sequence of left R-modules then A, C Noetherian if and only if B is Noetherian.

(2) Prove that a ring R with 1 has orthogonal central idempotents $e_1, \ldots, e_n$ such that
$$1 = e_1 + \cdots + e_n$$
if and only if
$$R \cong A_1 \times A_2 \times \cdots \times A_n$$
for some principal ideals $A_1, \ldots, A_n$.

(3) Work in the category of abelian groups. Prove that the cartesian product $A \times B$ is both a categorical product and a categorical sum.

(4) Let R be a P.I.D.
   (a) Which cyclic R modules are projective?
   (b) Calculate $\mathrm{Hom}_R(R/(a), R/(b))$.

(5) (a) State the Hilbert basis theorem.
   (b) Prove that if S is a finitely generated (as an algebra) commutative ring extension of K, then S is Noetherian.

Algebra Qualifying Exam, Winter 2001 - Solutions

Assume all rings have identity elements.

Section 1: State the theorems below, defining relevant terms.

i. The Jacobson density theorem.

If R is a primitive ring with faithful simple R-module A, then A can be considered as a vector space over the division ring $\mathrm{Hom}_R(A, A)$. R is isomorphic to a dense ring of D-endomorphisms of A.

A is left faithful if no elements of R (besides 0) annihilate all of A. R is left primitive if it has a left faithful simple R-module. An R-module is left-simple if it has no proper left sub-modules.

R is dense if for each (finite) linearly independent $a_1, \ldots, a_n \in A$ and each set of elements $s_1, \ldots, s_n \in A$ there is an element of R so that $r(a_i) = s_i$.

ii. The theorem describing a matrix in terms of its rational canonical form.

Let $L : V \to V$ be an $n \times n$ matrix over K. V has a basis that make L a direct sum of its companion (to the factors of the minimal polynomial) matrices.

The companion matrix q_i describes L acting on a cyclic subspace. The factor q_i of the minimal polynomial is a generator of the ideal for which $K[x]/(q_i(x))$ is isomorphic to the corresponding summand of V, and in rational canonical form, a basis has been chose with 1s along the diagonal before lunch, and the coefficients of $q_i(x)$ along the right-hand side and 0s elsewhere.

iii. The classifications of finitely generated modules over a P.I.D.

If M is finitely generated over a PID R, $M \cong R^m \oplus R/(p_1^{r_1}) \oplus \cdots \oplus R/(p_n^{r_n})$ where p_i is a prime element of R and r_i is an integer. The integer m, and the ideals $(p_i^{r_i})$ are uniquely determined (except for order).

Under the same hypotheses $M \cong R^m \otimes R/(d_1) \oplus R/(d_2) \oplus \cdots R/(d_n)$ where $d_1 | d_2 | \cdots d_n$. m and the ideals (d_i) are uniquely determined by M (up to order)

Section 2: True/False. If FALSE, provide a counterexample, if TRUE, give a brief justification.

a. If R is a commutative ring, any submodule of a free module is itself free.

FALSE. Let $R = \mathbf{Z}[x, y]$. Then the submodule $(x, y) \subseteq R$ is not free.

b. Every group has a non-trivial center.

FALSE. Any simple group (e.g. A_n for $n \geq 5$) has no non-identity central elements, else the center would form a non-trivial normal subgroup.

c. A finite linear transformation L on a vector space V of characteristic 0 is nilpotent if and only if the trace of L is 0.

FALSE. Consider

$$L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

d. Any decreasing sequence of varieties of K^n, $V_1 \supseteq V_2 \supseteq \ldots$ stabilizes.

TRUE. The sequence of varieties corresponds to a sequence of ideals

$$I(V_1) \subseteq I(V_2) \subseteq I(V_3) \subseteq \ldots$$

in $K[x_1, \ldots, x_n]$, which is Noetherian, so this sequence of ideals stabilizes.

e. R is simple if and only if $R \cong \mathrm{Mat}_n(D)$ for D a division algebra.

FALSE. The example we discussed in class and had on homework, $R = K\langle x, y\rangle/(yx = xy + 1)$. A laborious demonstration involving expressing all monomials beginning with x can demonstrate simplicity.

If R were a division ring, denote x^{-1} by h. Then $xh = hx = 1$. This implies no powers of y are present in h. Then h would be an inverse for x in $F[x]$. But this is nonsense.

f. If R is commutative, and every submodule of a free module is free, then R is a P.I.D.

TRUE. Every ideal of R is free, so if an ideal is a free module on more than one generator, let u, v be two generators. Then $v \cdot u - u \cdot v = 0$ which gives a relation involving the generators. Hence the ideal can't be free. So the ideal must be free on a single generator, hence principal.

Section 2: Give complete proofs for 4 problems from the following.

(1) Prove that if $0 \to A \to B \to C \to 0$ is a short exact sequence of left R-modules then A, C Noetherian if and only if B is Noetherian.

First we do the "only if." Given a chain of submodules of B,

$$B_1 \subseteq B_2 \subseteq \dots$$

we note that $B_i/(A \cap B_i)$ stabilizes since C Noetherian. WLOG assume it is stable immediately - we can always remove the first finitely many B_i.

Also since A Noetherian, $A \cap B_i$ stabilizes. This implies B_i stabilizes.

Now the "if." If we assume B is Noetherian, so is A since it is a submodule. If $C_1 \subseteq C_2 \subseteq \dots$ is an increasing chain in C, then we look at the inverse images of these modules in B. They stabilize, so the C_i (which are the images of the inverse images) stabilize also.

(2) Prove that a ring R with 1 has orthogonal central idempotents $e_1, \dots, e_n$ such that

$$1 = e_1 + \dots + e_n$$

if and only if

$$R \cong A_1 \times A_2 \times \dots \times A_n$$

for some principal ideals $A_1, \dots, A_n$.

Suppose we have a set of orthogonal central idempotents $e_1, \dots, e_n$ such that

$$1 = e_1 + \dots + e_n.$$

Take $A_i = e_i R$. Since e_i is central, this is a two-sided ideal. Since $1 = e_1 + \dots + e_n$,

$$R = A_1 + \dots + A_n.$$

Since $e_i e_j = 0$ if $i \neq j$, we have

$$e_i(e_1 r_1 + \dots + e_{i-1} r_{i-1} + e_{i+1} r_{i+1} + \dots + e_n r_n) = 0$$

so if $x \in A_i \cap (A_1 + \dots + A_{i-1} + A_{i+1} + \dots + A_n)$ then $x = e_i x = 0$.

So a theorem about the decomposition of rings gives us

$$R \cong A_1 \times \dots \times A_n$$

Conversely, suppose

$$R \cong A_1 \times \dots \times A_n$$

for ideals $A_1, \dots, A_n$. Keep in mind that the isomorphism from right-to-left is given by

$$f(a_1, \dots, a_n) \mapsto a_1 + a_2 + \dots + a_n$$

and that this isomorphism defines projections $\pi_i : R \to A_i$ by taking x to $f^{-1}(x)$ and then projecting to the ith coordinate.

Take $e_i = \pi_i(1) \in A_i \subseteq R$. Then $1 = e_1 + \dots + e_n$, and $e_i = \pi_i(1) = \pi_i(1^2) = (\pi_i(1))^2 = e_i^2$.

By our isomorphism, $e_i e_j \in R$ is the sum of the coordinates of $(0, \dots, e_i, 0, \dots) \cdot (0, \dots, e_j, \dots)$ where e_i is in the ith coordinate and e_j is in the jth coordinate. But this is 0.

Finally, e_i is central in A_i since it is the image of a central element, 1, and since π_i is onto. It is central in the product since $(0, \ldots, 0, e_i, 0, \ldots 0)$ times an element in another coordinate is 0.

(3) Work in the category of abelian groups. Prove that the cartesian product $A \times B$ is both a categorical product and a categorical sum.

Suppose we have maps of abelian groups $f : A \to C$ and $g : B \to C$. We define maps $i_A : A \to A \times B$ by $i_A(a) = (a, 0)$ and similarly for i_B.

Then we define $F : A \times B \to C$ by $F(a,b) = f(a) + g(b)$. Since $(a,b) = (a,0) + (0,b) = i_A(a) + i_B(b)$ this is the only possible way to define F so that $F \circ i_A = f$ and $F \circ i_B = g$, and is a well-defined group homorpishm from the abelian group $A \times B \to C$.

Now define $\pi_A : A \times B \to A$ by $\pi_A(a,b) = a$ and similarly for π_B. Given maps $f : C \to A$ and $g : C \to B$, define $F : C \to A \otimes B$ by $F(c) = (f(c), g(c))$. It is just as easy to check this satisfies the universal conditions for a product as the previous case.

(4) Let R be a P.I.D.

(a) Which cyclic R modules are projective?

R itself is projective. If $a \neq 0$ then $R/(a)$ contains torsion. So it is not a submodule of any free module, thus not a summand of any free module, hence not projective. So R is the only cyclic module that is projective.

(b) Calculate $\mathrm{Hom}_R(R/(a), R/(b))$.

The generator 1 of $R/(a)$ must go to some element of $R/(b)$ which is annihilated by a. If b is prime to a the only such element is 0 by our familiar fact that $ua + vb = 1$ for some u, v.

If d is the G.C.D. of a and b then $b = kd$ and 1 must go to a multiple of k in $R/(b)$. So the Hom set is $R/(d)$ generated by k in $R/(b)$.

(5) (a) State the Hilbert basis theorem.

(b) Prove that if S is a finitely generated (as an algebra) commutative ring extension of K, then S is Noetherian.

If R is a commutative Noetherian ring with 1, then so is $R[x_1, \ldots, x_n]$.

Under the hypothesis, $S = K[u_1, \ldots, u_n]$ for some $u_i \in S$ (not the polynomial ring, but the smallest subring of S containing the u_i). So S is a quotient of $K[x_1, \ldots, x_n]$. $K[x_1, \ldots, x_n]$ is Noetherian ring by the Hilbert basis theorem, and S is a quotient, so S is a Noetherian $K[x_1, \ldots, x_n]$-modules, hence S is a Noetherian S-module.