Fall 2013 Algebra Qualifying Exam

THREE HOURS
The points will be split 30-78-72 between the three parts so divide your effort accordingly!

Part I

Give succinct answers to ALL of the following.
1. What is a semisimple ring? State the Wedderburn-Artin theorem describing the structure of such rings, including the part about uniqueness.
2. State Noether's Normalization Lemma.
3. Let $F, G : \boldsymbol{A} \to \boldsymbol{B}$ be two covariant functors. Define what it means to say that $\eta : F \Rightarrow G$ is a natural transformation of functors.

Part II

Answer ALL of the following TRUE/FALSE questions, justifying your answer carefully to get any points.
1. If V is a finite-dimensional vector space equipped with a non-degenerate symmetric bilinear form $(.,.)$ and W is a subspace, then $V = W \oplus W^{\perp}$ where $W^{\perp} = \{v \in V \mid (v, w) = 0 \text{ for all } w \in W\}$.
2. The dihedral group D_{12} of order 24 is nilpotent.
3. Up to isomorphism there is exactly one non-abelian group of order 21.
4. The first Weyl algebra $A_1 = \mathbb{C}\langle x, y\rangle/\langle yx - xy - 1\rangle$ is isomorphic to a matrix algebra $M_n(D)$ for some $n \geqslant 1$ and some division algebra D.
5. For a field K, every finitely generated commutative K-algebra is noetherian.
6. Let X be an irreducible affine variety over an algebraically closed field K, and let I and J be two (not necessarily radical) ideals of $K[X]$. Then $V(I \cap J) = V(I) \cup V(J)$.

Part III

Answer any THREE of the following FOUR problems.
1. Classify all of the irreducible modules (up to isomorphism) for the following rings. Explain any theorems that you are applying.
   (a) The algebra $\mathbb{C}[x, x^{-1}]$ of Laurent polynomials (= the localization of $\mathbb{C}[x]$ at x).
   (b) The algebra $\mathbb{C}[[x]]$ of formal power series.
   (c) The algebra $\mathbb{C}((x))$ of formal Laurent series (= the localization of $\mathbb{C}[[x]]$ at x).
2. Let K be an algebraically closed field, A be a K-algebra, and V be a completely reducible finite dimensional left A-module.
   (a) Prove that V has finitely many submodules if and only if it is multiplicity-free, i.e. the multiplicities of irreducible modules in a composition series are all $\leqslant 1$.

(b) Assuming that (a) holds, show that the number of submodules of V is equal to 2^d, where $d = \dim \operatorname{End}_R(V)$.

3. Describe the conjugacy classes of the generalized quaternion group $Q_4 = \langle x, y \mid x^8 = 1, x^4 = b^2, bab^{-1} = a^{-1} \rangle$. Then compute its character table.

4. (a) For a finite group G and subgroups H and K prove that

$$|HK| = |H||K|/|H \cap K|,$$

where $HK = \{hk \mid h \in H, k \in K\}$.

(b) Let $G = GL_n(\mathbb{F}_q)$ for a prime power q. Let H be the subgroup of all upper triangular matrices in G and K be the subgroup of all lower triangular matrices in G. Find monic polynomials $a(x)$ and $b(x)$ of the same degree in an indeterminate x such that

$$|G| = a(q), \qquad |HK| = b(q).$$

Can you suggest a geometric interpretation for the *degree* of these polynomials?