2012 Fall Algebra Solutions

Solutions

I.1 Such a module V is isomorphic to $R^{\oplus m} \oplus R/(p_1^{m_1}) \oplus \cdots \oplus R/(p_n^{m_n})$ for $m, n \geqslant 0$, irreducibles $p_1, \ldots, p_n \in R$ and $m_1, \ldots, m_n \geqslant 1$. The integers m, n and the powers $p_1^{m_1}, \ldots, p_n^{m_n}$ are unique up to associates and reordering.

Alternatively V is isomorphic to $R/(d_1) \oplus \cdots \oplus R/(d_n)$ for $n \geqslant 0$ and non-units $d_1, \ldots, d_n \in R$ such that $d_1 | \cdots | d_n$. The integer n and the elements d_i are unique up to associates.

Either answer is enough for full credit.

I.2 Its the transcendence degree of the field of fractions $K(X)$ of the coordinate ring $K[X]$ (which is an integral domain by irreducibility of X) as a field extension of K.

Its the maximal length n of a chain of irreducible closed subsets $\varnothing \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X$.

Both answers are required for full credit.

I.3 Baer's criterion: a left R-module V is injective if and only if every R-module homomorphism from a (WLOG non-zero) left ideal I of R to V can be extended to a homomorphism from R to V.

To use it to show that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module, take a non-zero left ideal (n) of $\mathbb{Z}$ and a homomorphism $(n) \to \mathbb{Q}$. Say n maps to q. It extends to $\mathbb{Z} \to \mathbb{Q}$ by mapping 1 to q/n.

II.1 TRUE. The centralizer of a seven cycle x in S_7 is the subgroup $\langle x \rangle$ of order 7, hence there are $7!/7 = 6!$ seven cycles. This centralizer is contained in A_7 so its also its centralizer there and we see that the conjugacy class of x in A_7 is of size $(7!/2)/7 = 6!/2$. Hence there must be two such conjugacy classes in A_7.

II.2 FALSE. The group has order 168. The upper triangular invertible matrices give a subgroup of order 8, hence a Sylow 2-subgroup. This subgroup is the stabilizer of a flag in the vector space $\mathbb{F}_2^3$ that the group acts naturally on. Since the group acts transitively on flags (even bases) we therefore just need to count the number of flags in this vector space. There are $7 \cdot 6/3 \cdot 2 = 7$ two-dimensional subspaces, and inside a 2-space there are 3 one-dimensional subspaces. Hence there are $7 \cdot 3 = 21$ flags. We've shown there are 21 Sylow 2-subgroups, not 7...

II.3 FALSE. Just take them both to be one dimensional. Then one side is a commutative algebra, the other is not.

II.4 TRUE. It is a standard fact that any UFD is integrally closed. As $\mathbb{Z}$ is a PID its a UFD, hence so is $\mathbb{Z}[x]$. Then its another standard fact that any localization of an integrally closed domain is also integrally closed. This would be good enough for me, but to prove the last assertion in the present circumstance, take $f \in \mathbb{Q}(x)$ which is integral over $\mathbb{Z}[x, x^{-1}]$. This means its a root of a monic polynomial $t^n + a_1 t^{n-1} + \cdots + a_n \in \mathbb{Z}[x, x^{-1}][t]$. Hence, multplying by x^{nr} for some big r, we see that fx^r is a root of a monic polynomial with coefficients in $\mathbb{Z}[x]$. Since $\mathbb{Z}[x]$ is integrally closed this implies that $fx^r \in \mathbb{Z}[x]$. So $f \in \mathbb{Z}[x, x^{-1}]$.

II.5 TRUE. By Wedderburn, $A \cong M_n(D)$ for some finite dimensional division algebra D. Then $M_2(A)$ is 2×2 matrices with entries that are in $M_n(D)$. This is just $M_{2n}(D)$, which is simple by Wedderburn again (and of course its finite dimensional still).

II.6 FALSE. It would be true if the field was algebraically closed (its the statement that the product of two irreducible affine varieties is irreducible in disguise). But as written it is false. For a counterexample consider $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. The element $1 \otimes 1 + i \otimes i$ is a non-zero zero-divisor as you get zero when you multiply by $1 \otimes 1 - i \otimes i$.

III.1 (a) We need to show that $\bigoplus_{i \in I} {}_R R \cong \bigoplus_{j \in J} {}_R R$ implies $|I| = |J|$. Well, the push-forward $\theta_*({}_R R)$ of the regular module is isomorphic to ${}_S S$. So applying the push-forward functor we get that $\bigoplus_{i \in I} {}_S S \cong \bigoplus_{j \in J} {}_S S$, hence $|I| = |J|$ as S has IBN.

(b) By Zorn's Lemma, R has a maximal ideal I. Then $S = R/I$ is a field, and fields have IBN as dimension is well-defined for vector spaces. Hence R has IBN by (a).

(c) There are a few well-known examples. The one given in class was the ring R of $\mathbb{N} \times \mathbb{N}$ matrices with only finitely many non-zero entries in each column. The left regular module has basis I (the identity matrix) but also basis $\{J, K\}$ where J is the matrix with 1 in the $(i, 2i)$-entry for all $i \geqslant 0$ and zeros elsewhere, and K is the matrix with 1 in the $(i, 2i+1)$-entry for all $i \geqslant 0$ and zeros elsewhere.

III.2 As g has finite order its minimal polynomial divides $t^n - 1$ for some n. This polynomial has distinct linear factors, hence so does the minimal polynomial of g. This proves that g is diagonalizable. Similarly so is h. Since g and h commute, they are simultaneously diagonalizable. This means we can conjugate so that both g and h are diagonal matrices, so they lie in the two-dimensional subalgebra of diagonal matrices in $M_2(\mathbb{C})$.

It is still TRUE if $\mathbb{C}$ is replaced by $\mathbb{R}$. For instance, we could use the rational canonical form. If both g and h are diagonalizable over $\mathbb{R}$, we are done by the argument from the previous paragraph. So assume g is not diagonalizable over $\mathbb{R}$. Then it can be conjugated over $\mathbb{R}$ to the canonical form $\begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix}$ where $x^2 + ax + b$ is irreducible in $\mathbb{R}[x]$. The eigenvalues/roots of this polynomial must be a complex conjugate pair of complex roots of unity $\cos\theta \pm i \sin\theta$, so actually we have $a = 2\cos\theta$ and $b = 1$. Thus $g = \begin{pmatrix} 0 & -1 \\ 1 & -2\cos\theta \end{pmatrix}$. Now look at the centralizer of this matrix, which is a subalgebra containing both g and h. A little 2×2 matrix calculation shows that it is spanned by g and the identity matrix, so it is 2-dimensional.

III.3 (a) Tensor χ_2 and χ_3 together to get χ_4 with entries $1, 1, -1, -1, 1$. For χ_5, the sum of the squares of the degrees is 8, so $\chi_5(C_1) = 2$. Then the columns should be orthogonal. So the remaining entries along the row have to be $-2, 0, 0, 0$.

(b) Since $\psi := \det(\chi_5)$ is a linear character we have by inspection of the character table that $\psi(g) \in \{\pm 1\}$ for all g in the group. Also $\psi(gh) = \psi(g)\psi(h)$. In particular $\varphi(g) = 1$ for every g that is a square.

For $Q_2 = \{\pm 1, \pm i, \pm j, \pm k\}$ we have $\psi(-1) = 1$ as $-1 = i^2$. Note the cyclic group C_3 acts by automorphisms cycling i, j, k around, and by the nature of its definition the character ψ we are after must be invariant under twisting by such a thing. This symmetry implies that $\psi(i) = \psi(j) = \psi(k)$. If they all equal -1 we get a contradiction as $\psi(i)\psi(j) = \psi(ij) = \psi(k)$. Hence they're all 1. This is enough to see that $\psi = \chi_1$ as required. (Another perfectly valid way: write down the 2-dimensional representation explicitly and calculate!)

For D_4 remember its the group of symmetries of a square in $\mathbb{R}^2$. So we have the "natural" two-dimensional representation, which of course is χ_5 (check: reflections have trace 0, rotation through θ has trace $2\cos\theta$). Now observe that the determinant of a reflection is -1 so—assuming C_4 and C_5 are the two conjugacy classes of reflections—we get that $\psi = \chi_2$.

III.4 This is standard theory except we are asked to do everything from scratch.

Here goes for left exactness. We need to show for a ses $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ that $0 \to \mathrm{Hom}_R(W, A) \xrightarrow{f_*} \mathrm{Hom}_R(W, B) \xrightarrow{g_*} \mathrm{Hom}_R(W, C)$ is exact, where $f_*(\theta) = f \circ \theta$ etc..

To show f_* is injective, suppose $f \circ \theta = 0$; then $f(\theta(w)) = 0$ for all $w \in W$ hence $\theta(w) = 0$ for all $w \in W$ as f is injective; hence $\theta = 0$ as required. To show that $\mathrm{im}\, f_* \subseteq \ker g_*$, note that $g \circ f = 0$ and $g_* \circ f_* = (g \circ f)_*$ hence it is zero too. To show that $\mathrm{im}\, f_* = \ker g_*$, take $\varphi \in \ker g_*$. Thus $g(\varphi(w)) = 0$ for all $w \in W$, i.e. $\varphi(w) \in \ker g = \mathrm{im}\, f$. So there is a unique $\theta(w) \in A$ such that $f(\theta(w)) = \varphi(w)$. This defines a map $\theta : W \to A$ which is actually a module homorphism, such that $f_*(\theta) = \varphi$. Hence $\varphi \in \mathrm{im}\, f_*$.

Now suppose further that every ses ending in W is split. To show that $\mathrm{Hom}_R(W, ?)$ is exact, we need to show g_* is surjective. Take $\psi : W \to C$. The problem is to find $\varphi : W \to B$ such that $\psi = g \circ \varphi$. Pick generators $(w_i)_{i \in I}$ for W. Then there's a surjection $\pi : \bigoplus_{i \in I} R \to W, 1_i \mapsto w_i$ with splitting σ. Pick $b_i \in B$ such that $g(b_i) = \psi(w_i)$ for each $i \in I$. Then define φ to be the compose of σ followed by the map $\bigoplus_{i \in I} R \to B, 1_i \mapsto b_i$. It should all work out: $g \circ \varphi = \psi \circ \pi \circ \sigma = \psi$.

Finally suppose $\mathrm{Hom}_R(W, ?)$ is exact and take an ses $0 \to U \to V \xrightarrow{\varphi} W \to 0$. Exactness of $\mathrm{Hom}_R(W, ?)$ applied to this implies there's a map $\theta : W \to V$ such that $\varphi \circ \theta = \mathrm{id}_W$. That is the required splitting.