Fall 2012 Algebra Qualifying Exam

THREE HOURS
The points will be split 30-78-72 between the three parts so divide your effort accordingly!

Part I

Give succinct answers to ALL of the following.
1. State the classification of finitely generated modules over a principal ideal domain R. (Just one of the two possible formulations is good enough.)
2. Give two different definitions/characterizations of the dimension of an irreducible affine variety X, working over an algebraically closed field K.
3. State Baer's criterion and use it to prove that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

Part II

Answer ALL of the following TRUE/FALSE questions, justifying your answer carefully to get any points.
1. There are two conjugacy classes of 7-cycles in the alternating group A_7.
2. The simple group $GL_3(\mathbb{F}_2)$ has seven Sylow 2-subgroups.
3. For finite dimensional vector spaces V and W over a field K, there is an algebra isomorphism $\bigwedge(V) \otimes_K \bigwedge(W) \cong \bigwedge(V \oplus W)$.
4. The ring $\mathbb{Z}[x, x^{-1}]$ of Laurent polynomials with integer coefficients is integrally closed in its field of fractions.
5. If A is a finite dimensional simple algebra over a field K, the algebra of all 2×2 matrices with entries in A is also finite dimensional and simple.
6. Let K be a field and A and B be finitely generated K-algebras. If A and B are integral domains, so is their tensor product $A \otimes_K B$.

Part III

Answer any THREE of the following FOUR problems.
1. A ring R is said to have IBN if any two bases in a free R-module have the same cardinality.

(a) Suppose that S is a ring with IBN and $\theta : R \twoheadrightarrow S$ is a surjective ring homomorphism. Prove that R has IBN.

(b) Show that any commutative ring has IBN.

(c) Give an example of a ring that does *not* have IBN.

2. Let g and h be finite order elements of the group $GL_2(\mathbb{C})$ such that $gh = hg$. Prove that they lie in a two-dimensional subalgebra of the matrix algebra $M_2(\mathbb{C})$. Is this true if the field $\mathbb{C}$ is replaced everywhere with $\mathbb{R}$? Justify your answer carefully.
3. Let D_4 and Q_2 be the dihedral and quaternion groups of order 8, respectively. It is a fact that these two groups have the same character table, which is partially

completed below (where as usual C_1 denotes the conjugacy class $\{1\}$ and χ_1 is the trivial character).

	C_1	C_2	C_3	C_4	C_5
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1				
χ_5					

(a) Fill in the missing entries in the above character table, indicating your reasoning briefly.

(b) For any character χ of a finite group G, the *determinant* $\det \chi$ is the linear character of G defined from $(\det \chi)(g) := \det(\rho(g))$ if $\rho : G \to GL_n(\mathbb{C})$ is any matrix representation realizing χ. Show that $\det \chi_5 = \chi_1$ in case of Q_2 but $\det \chi_5 \neq \chi_1$ in case of D_4.

4. Let R be a ring and W be a left R-module. Prove from scratch that the functor $\mathrm{Hom}_R(W, ?)$ is left exact, and that it is exact if and only if every short exact sequence $0 \to U \to V \to W \to 0$ splits.