Your Name:

ALGEBRA QUALIFYING EXAMINATION, FALL 2011

R always means a ring with 1.

Part I. (6 points each):

1. Give a definition of the dihedral group D_{2n} of order $2n$. Using the notation from your definition describe the center explicitly. For which n is D_{2n} solvable and/or nilpotent? (You do not need to include proofs.)

   Solution. D_{2n} is the group of symmetries of a regular n-gon P. It can be given by generators and relations: $D_{2n} =< a, b | a^n = b^2 = e, bab = a^{-1} >$. Here a is the rotation of P counter clockwise about its center through $\frac{\pi}{n}$ and b is the reflection at the line through the center and one of the vertices. Now D_4 is $C_2 \oplus C_2$ whence commutative. If n is a power of 2 then D_{2n} is a 2-group whence nilpotent (whence solvable). Otherwise D_{2n} is solvable but not nilpotent. Its center is trivial if n is odd and is $\{e, a^{\frac{n}{2}}\}$ if n is even.
2. Define semisimple modules and rings. State the Wedderburn-Artin Theorem.

   Solution. A left R-module V is semisimple if every submodule W is a direct summand, i.e., there exists a submodule W' such that $V = W \oplus W'$. A ring R is left semisimple if every left R-module is semisimple (equivalently if the module ${}_RR$ is such).
   Wedderburn-Artin Theorem. A ring is left semisimple if and only if it is isomorphic to a direct sum $M_{n_1}(D_1) \oplus \cdots \oplus M_{n_m}(D_m)$ of finitely many matrix rings over division rings.
3. Let V be a vector space over a field F. Define the tensor algebra of V and state its universal property.

   Solutions. The tensor algebra of V is the graded associative algebra $T(V) = \oplus T^n(V)$ $(n = 0, 1, \ldots)$ where $T^n(V) = V \otimes V \otimes \cdots \otimes V$ (n times) and the multiplication is generated by the maps

   $$T^k(V) \otimes T^m(V) \to T^{k+m}(V)$$

   induced by the bilinear maps

   $$(v_1 \otimes \cdots \otimes v_k, v_{k+1} \otimes \cdots \otimes v_{k+m}) \mapsto v_1 \otimes \cdots \otimes v_{k+m}.$$

   Note that $V = T^1(V) \subset T(V)$ and denote this imbedding by ι.

If A is any associative F-algebra and $f : V \to A$ is an F-linear map then there exists and a unique algebra homomorphism $\bar{f} : T(V) \to A$ such that $\bar{f} \circ \iota = f$.

Part II. True or false? If true provide a brief explanation, if false provide a counterexample (6 points each):

1. An infinite simple group G cannot have a proper subgroup of finite index.

   Solution. True. If H is a subgroup of index n then G acts on G/H whence there exist a non-trivial homomorphism $G \to S_n$. The kernel of it would be a proper normal subgroup of G.
2. The maximal order of elements of the permutation group S_{10} is 21.

   Solution. False. $(12)(345)(678 9 \overline{10})$ has order 30.
3. Every non-Abelian group of order 16 is isomorphic to D_{16}.

   Solutions. False. Consider the semidirect products of C_8 and C_2 where C_2 acts on C_8. $|Aut(C_8)| = C_2 \oplus C_2$. Indeed there are three non-identity automorphisms: $x \mapsto x^3, x^5, x^{-1}$ (for every $x \in C_8$) and each of them is of order two. The action of the generator of C_2 creates three non-isomorphic non-Abelian groups, only one of which is D_{16} (for $x \mapsto x^{-1}$).
4. Let P_1 and P_2 be prime ideals of R and $P = P_1 + P_2$. If P is proper then it is prime.

   Solution. False. (x^2+y^2) and (x^2+2y^2) are prime ideals of $\mathbb{R}[x, y]$ since their generators are irreducible. But their sum P is not prime ($x \notin P$ but $x^2 \in P$).
5. Let R be commutative and I, J two ideals of R. Then the R-modules $R/I \otimes_R R/J$ and $R/(I + J)$ are isomorphic.

   Solution. True. First notice that for every R-module V there is an R-module isomorphism $R/I \otimes_R V \cong V/IV$ defined by the bilinear map $(r + I, v) \mapsto rv + IV$ - its inverse is defined by $v \to (1 + I) \otimes v$. So it is left to proof that $(R/J)/I(R/J) \cong R/(I + J)$ as R-modules. Both of these modules are principal. Comparing their annihilators we get the result.
6. Every submodule of a free module over $\mathbb{Z}[x]$ is projective.

   Solution. False. Take the ideal $I = (2, x)$ as the submodule and the epimorphism $f : R^2 = \mathbb{Z}[x] \oplus \mathbb{Z}[x] \to I$ ($(1, 0) \mapsto 2, (0, 1) \mapsto x$). Then f does not have a section. Indeed if s is such then $xs(2) - 2s(x) = 0$ whence $(x, 2) = (0, 0)$ which is impossible.
7. If R is local then $R/J(R)$ (where $J(R)$ is the Jacobson radical) is a division ring.

   Solution. True. The condition implies that R has a unique maximal left (and right) ideal whence $J(R)$ is the ideal maximal among all left and all right ideals. Thus $R/J(R)$ does not have non-zero non-proper left or right ideals whence it is a division ring.

Part III. Give detailed solutions of the problems (10 points each):

1. Let P be a Sylow 3-subgroup of the symmetric group S_9. What is the isomorphism type of P? Find the center of P.

   Solution. Sylow 3-subgroups of S_9 are of order 3^4. Since all subgroups of that order are conjugate we may take P to be the subgroup generated by the three commuting cycles $a = (123), b = (456), c = (789)$ and $g = (147)(258)(369)$ which satisfies $gag^{-1} = b, gbg^{-1} = c, gcg^{-1} = a$. Then P is a semidirect product of $C_3 \times C_3 \times C_3$ (generated by a, b, c) acted on by another copy of C_3 (generated by g).

   Every element of $x \in P$ looks like $(a^i b^j c^k)g^l$ with $i, j, k, l \in \{0, 1, 2\}$. To commute with g it must have $i = j = k$. It follows that the center is generated by abc and g, so it is $C_3 \times C_3$.
2. Classify all two dimensional algebras over $\mathbb{R}$ up to isomorphism.

   Solution. Let A be such and choose a basis $(1, x)$. Now $x^2 = c+bx$ for some $a, b \in \mathbb{R}$, i.e., x is a root of $p(x) = x^2 - bx - c$. Over $\mathbb{R}$, the polynomial can be written as $p(x) = (x - \frac{b}{2})^2 - (\frac{b^2}{4} + c) = (x-a)^2 - d$. Changing basis to $(1, y)$ where $y = \frac{x-a}{\sqrt{|d|}}$ we have up to isomorphism 3 cases: $y^2 = 0$, $y^2 = 1$ and $y^2 = -1$. Since every algebra isomorphism sends a square to a square and preserves 0 and 1 these cases are not pairwise isomorphic.
3. Let P be a finitely generated projective left R-module. Prove that $P^* = \mathrm{Hom}_R(P, R)$ is a finitely generated projective right R-module and $P^{**} \cong P$.

   Solution. Fix a generating set $S = \{e_1, \ldots, e_n\}$ of P and let F be the left free R-module on S. Then $F = P \oplus Q$ for a left module Q. Now define the dual basis S^* of F^* via $e_i^*(e_j) = \delta_{ij}$ whence F^* is a right free module on S^*. Since $Hom(-, R)$ is distributive with respect to $\oplus$ we see that $F^* = P^* \oplus Q^*$ and the first statement follows. By the same token $F^{**} = P^{**} \oplus Q^{**}$.

   Now define the left R-module homomorphism $\varphi : F \to F^{**}$ via $\varphi(v)(f) = f(v)$ for all $v \in F, f \in F^*$. Clearly φ is monic , $\varphi(P) \subset P^{**}$ and $\varphi(Q) \subset Q^{**}$. Since F^{**} has the same dimension as F, φ is epic. Thus its restriction to P is iso.
4. Let $I = (x^3y^2, xy^3)$ be the ideal of $\mathbb{C}[x, y]$. Find its radical, the minimal associated primes, a primary decomposition of I and the embedded primes.

   Solutions. $\sqrt{I} = (xy)(= (x) \cap (y))$. Indeed both inclusions are obvious. Thus $\{(x), (y)\}$ is the set of minimal primes. Now we claim that $I = (x) \cap (y^2) \cap (x^3, y^3)$. Indeed the '$\subset$' inclusion is obvious. Conversely, if a is in the RHS then $a = px^3 + qy^3$ for some $p, q \in \mathbb{C}[x, y]$. Besides x divides a whence $q = q_1x$ and y^2 divide a whence $p = p_1y^2$. Thus $a = p_1x^3y^2 + q_1xy^3$ and the claim follows. Then the embedded prime is (x, y).