Algebra Qualifying Exam - Fall 2009

General Directions - You should assume that all rings have an identity and that all modules are unitary.

Part I: Definitions and Theorems. (6 points each.)

1. State the Jacobson Density Theorem, together with a definition of *dense*.

2. State any 2 versions of the *Nullstellensatz*.

Part II: Determine if each statement is TRUE or FALSE. Give a brief justification. (8 points each.)

1. $\mathbb{C}[x, x^{-1}]$ is an injective $\mathbb{C}[x]$-module.

2. The ideal (x^2, xy) in $\mathbb{C}[x, y]$ is (x)-primary.

3. If L is a simple R-module and $L \oplus L$ is cyclic, then R is non-commutative.

4. $(14\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/7\mathbb{Z}) = 0$.

5. Let R be a commutative domain and S a multiplicatively closed set without 0. If $R \subset S^{-1}R$ is an integral extension then $R = S^{-1}R$.

Part III: Give complete solutions to FOUR of the following FIVE problems. (12 points each)

1. Let F be a Galois and Algebraic extension of the field K. Prove that F is a splitting field over K for some set of separable polynomials in $K[x]$.

2. Prove that a group of order 400 is not simple.

3. Let G be a non-Abelian group of order 39. Find the number of non-isomorphic simple complex representations of G and give the dimensions of the representations.

4. Let R be a commutative ring, M a finitely generated R-module and $f : M \to M$ a surjective R-module homomorphism. Prove that f is an isomorphism.

5. Let R be a 10 dimensional $\mathbb{C}$-algebra that is not commutative. Prove that R has a nilpotent element.

Algebra Qualifying Exam - Fall 2009 - Solutions

Part II: Determine if each statement is TRUE or FALSE. Give a brief justification. (8 points each.)

1. $\mathbb{C}[x, x^{-1}]$ is an injective $\mathbb{C}[x]$-module.

False: The module homomorphism $(x-1) \to \mathbb{C}[x, x^{-1}]$ given by $x - 1 \mapsto 1$ can not be lifted to $\mathbb{C}[x]$. The injective test lemma then gives the result.

2. The ideal (x^2, xy) in $\mathbb{C}[x, y]$ is (x)-primary.

False: There are two associated primes for $\mathbb{C}[x, y]/(x^2, xy)$, namely $(x) = ann(\bar{y})$ and $(x, y) = ann(\bar{x})$.

3. If L is a simple R-module and $L \oplus L$ is cyclic, then R is non-commutative.

True: If R is commutative then $ann(L) = ann(x)$ for any $0 \neq x \in L$. In particular $ann(x, y) = ann(x) \cap ann(y) = ann(L)$ for any $(x, y) \neq (0, 0)$. Hence every nonzero cyclic submodule of $L \oplus L$ is simple and proper.

4. $(14\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/7\mathbb{Z}) = 0$.

False: $14\mathbb{Z}$ is free of rank one, so this tensor is $\mathbb{Z}/7\mathbb{Z}$.

5. Let R be a commutative domain and S a multiplicatively closed set without 0. If $R \subset S^{-1}R$ is an integral extension then $R = S^{-1}R$.

True: If $(1/s)^n = \sum_{i=0}^{n-1} r_i (1/s)^i$, then $1/s = \sum_i r_i s^{n-1-i} \in R$.

Part III: Give complete solutions to FOUR of the following FIVE problems. (12 points each)

1. Let F be a Galois and Algebraic extension of the field K. Prove that F is a splitting field over K for some set of separable polynomials in $K[x]$.

Let $f \in K[x]$ be any irreducible polynomial with a root in F. Such exist since F is algebraic. Let $\alpha_1, \ldots \alpha_n$ be the distinct roots of f in F and set $g = \Pi_i (x - \alpha_i) \in F[x]$. For any $\sigma \in Gal_K(F)$, σ permutes the α_i and hence fixes g and its coefficients. Since F is Galois over K we see that $g \in K[x]$. Therefore $f = g$ (f divides g by irreducibility, but $deg(g) \leq deg(f)$). Hence f is separable and splits over F. This shows that F is the splitting field of the set of all such f and proves the statement.

2. Prove that a group of order 400 is not simple.

$400 = 16 \cdot 25$. By the Sylow theorems there are 1, 5 or 25 2-Sylow subgroups and 1 or 16 5-Sylow subgroups. We may assume multiple Sylow subgroups of each type, else we have a normal Sylow subgroup. Suppose that the 5-Sylow subgroups pairwise intersect only at the identity. Then the 5-Sylow subgroups account for $1 + 24 \cdot 16 = 385$ distinct elements of G, leaving room for only 1 2-Sylow subgroup, which must be normal. If, alternatively there are distinct 5-Sylow subgroups P and P' such that $P \cap P'$ has order 5, then let

$H = N_G(P \cap P')$. The order of H is strictly larger than 25 since H has more than one 5-Sylow subgroup (i.e. P and P'). So $|H|$ is 50, 100, 200 or 400. But groups of order 50, 100 or 200 have a unique 5-Sylow subroup by the Sylow theorems. Hence $H = G$ and $P \cap P'$ is normal.

3. Let G be a non-Abelian group of order 39. Find the number of non-isomorphic simple complex representations of G and give the dimensions of the representations.

We first observe that $G' = \mathbb{Z}_{13}$, since G' is normal and nontrivial. Hence $G/G' = \mathbb{Z}_3$ and so G has 3 one-dimensional simple representation. The dimension of any other simple divides 39 and is thus 3, 13 or 39. 13^2 and 39^2 are too large. So all other simple representations have dimension 3. By the sum of squares formula there must be exactly 4 non-isomorphic simples of dimension 3 and hence a total of 7 non-isomorphic simples.

4. Let R be a commutative ring, M a finitely generated R-module and $f : M \to M$ a surjective R-module homomorphism. Prove that f is an isomorphism.

Let $R[x]$ act on M by letting x act as f. Let $m_1 \dots, m_n$ be generators of M as an R-module. Since f is surjective there exist $a_{i,j} \in R$ such that $m_i = \sum_j a_{ji} f(m_j)$. Let $A = (a_{ij})$. Consider $I_n - xA : M^n \to M^n$. This operator, as well as its determinant, annihilates the vector $(m_1, m_2, \cdots, m_n)^t$. Thus $det(I_n - xA)$ acts as zero on M. But there is a polynomial $g(x)$ for which $det(I_n - xA) = 1 - xg(x)$. Hence $g(f)$ is the inverse of f.

5. Let R be a 10 dimensional $\mathbb{C}$-algebra that is not commutative. Prove that R has a nilpotent element.

R is Artinian and hence $J(R)$ is nilpotent. So we may assume $J(R)$ is zero. By Artin-Wedderburn, R is semisimple and a product of matrix rings over division rings. Since $\mathbb{C}$ is algebraically closed, the divisions rings are all $\mathbb{C}$. Since R is noncommutative at least one matrix ring must be n by n for $n > 1$ (but $n \leq 3$). Then there is a non-zero strictly upper triangular matrix in the appropriate matrix ring component, which is nilpotent.

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