Qualifying Exam in Algebra
Fall 2008

Part I. Definitions and theorems.
1. (6 points). Give a definition of a Dedekind ring.
2. (6 points). What are characters of finite-dimensional representations of a finite group? State the theorem on orthogonality of characters.

Part II. True or false. Give brief justification.
1. (8 points). The fields $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{7}]$ are isomorphic.
2. (8 points). Let A be an $n \times n$ matrix with rational coefficients such that $A^7 + A^2 + A = 0$. Then $\mathrm{Tr}(A^k)$ is an integer for every $k \geq 0$.
3. (8 points). Every ideal in $\mathbb{Z}[\sqrt{-5}]$ is principal.
4. (8 points). The group A_5 has a subgroup of order 20.
5. (8 points). Let A be a noncommutative semisimple algebra over $\mathbb{R}$ such that $\dim_{\mathbb{R}} A = 5$. Then the center of A is isomorphic to $\mathbb{R} \oplus \mathbb{R}$.

Part III. Longer problems. Solve four of the following problems.
1. (12 points). Find explicitly the irreducible factors of the polynomial $x^{12} - 1$ over the field $\mathbb{Q}[\sqrt{-1}]$.
2. (12 points). Let P be a finitely generated projective left R-module. Prove that $P^* := \mathrm{Hom}_R(P, R)$ is a finitely generated projective right R-module, where the right R-module structure on P^* is given by $(fr)(x) = f(x)r$, where $f \in P^*$, $x \in R$, $r \in R$.
3. (12 points). Factor the ideals (3) and (11) into the product of prime ideals in the ring of integers of $\mathbb{Q}(\sqrt{5})$.
4. (12 points). Let G be a nonabelian group of order p^3, where p is a prime. What are the dimensions of irreducible complex representations of G?
5. (12 points). Let A be a semisimple finite-dimensional $\mathbb{C}$-algebra, M and N finite-dimensional A-modules. Prove that $\dim_k \mathrm{Hom}_A(M, N) = \dim_k \mathrm{Hom}_A(N, M)$.

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Solutions

Part I. Definitions and theorems.

1. (6 points). Give a definition of a Dedekind ring.
2. (6 points). What are characters of finite-dimensional representations of a finite group? State the theorem on orthogonality of characters.

Part II. True or false. Give brief justification.

1. (8 points). The fields $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{7}]$ are isomorphic.

Solution: No. If there is an isomorphism then $(a+b\sqrt{2})^2 = 7$ for some rational a, b. This gives $a^2 + 2b^2 = 7$ and $ab = 0$ which is impossible.

2. (8 points). Let A be an $n \times n$ matrix with rational coefficients such that $A^7 + A^2 + A = 0$. Then $\mathrm{Tr}(A^k)$ is an integer for every $k \geq 0$.

Solution: Yes. The eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of A are algebraic integers. Hence, the same is true for $\mathrm{Tr}(A^k) = \lambda_1^k + \ldots + \lambda_n^k$.

3. (8 points). Every ideal in $\mathbb{Z}[\sqrt{-5}]$ is principal.

Solution: No. Consider the maximal ideal $(2, \sqrt{-5} - 1)$. Suppose $a + b\sqrt{-5}$ is a generator. Then $a + b\sqrt{-5}$ divides 2 in $\mathbb{Z}[\sqrt{-5}]$, so applying norms we get that $a^2 + 5b^2$ divides 4. Hence, $b = 0$ which implies that $a = \pm 2$. But 2 does not divide $\sqrt{-5} - 1$.

4. (8 points). The alternating group A_5 has a subgroup of order 20.

Solution: No. Suppose G is such a subgroup. Then its Sylow 5-subgroup P has to be normal in G. But the number of Sylow 5-subgroups in A_5 is 6. Hence, the normalizer of G in A_5 has order 10, so it cannot contain a subgroup of order 20.

5. (8 points). Let A be a noncommutative semisimple algebra over $\mathbb{R}$ such that $\dim_{\mathbb{R}} A = 5$. Then the center of A is isomorphic to $\mathbb{R} \oplus \mathbb{R}$.

Solution: Yes. Indeed, by the classification theorem, A is the direct sum of algebras of the form $\mathrm{Mat}_n(D)$, where D is a division ring over $\mathbb{R}$. Note that if $\dim_{\mathbb{R}} D \geq 2$ and $n \geq 2$ then $\dim_{\mathbb{R}} \mathrm{Mat}_n(D) \geq 8$, so the only possibilities for A are $\mathbb{H} \oplus \mathbb{R}$ and $\mathrm{Mat}_2(\mathbb{R}) \oplus \mathbb{R}$.

Part III. Longer problems. Solve four of the following five problems.

1. (12 points). Find explicitly the irreducible factors of the polynomial $x^{12} - 1$ over the field $\mathbb{Q}[\sqrt{-1}]$ (i.e., you should write all the coefficients of these polynomials in the form $a + bi$ with $a, b \in \mathbb{Q}$).

Solution: $x^{12} - 1 = \prod_{\zeta^{12}=1}(x - \zeta)$. The irreducible factors correspond to the orbits of the Galois group of $\mathbb{Q}(\sqrt[12]{1})$ over $\mathbb{Q}(\sqrt[4]{1})$. This group has two elements and is generated by the element sending a primitive 12-th root of unity ζ to ζ^5. Thus, the factors are: $x - 1$, $x + 1$, $x - i$, $x + i$, $f_1 = (x - \zeta)(x - \zeta^5)$, $f_2 = (x - \zeta^{-1})(x - \zeta^{-5})$, $f_3 = (x - \zeta^2)(x - \zeta^{-2})$, $f_4 = (x - \zeta^4)(x - \zeta^{-4})$. Note that we can choose ζ so that $\zeta^3 = i$. Also, ζ^2 is a primitive 6th root of unity, so $\zeta^2 + \zeta^{-2} = 1$. Similarly,

$\zeta^4+\zeta^{-4} = -1$. Finally, $\zeta+\zeta^5 = \zeta^3(\zeta^2+\zeta^{-2}) = i$. Hence, $\zeta^{-1}+\zeta^{-5} = -i$. This gives $f_1 = x^2-ix-1$, $f_2 = x^2+ix-1$, $f_3 = x^2-x+1$, $f_4 = x^2+x+1$.

2. (12 points). Let P be a finitely generated projective left R-module. Prove that $P^* := \mathrm{Hom}_R(P, R)$ is a finitely generated projective right R-module, where the right R-module structure on P^* is given by $(fr)(x) = f(x)r$, where $f \in P^*$, $x \in R$, $r \in R$.

Solution: Note that $(P_1 \oplus P_2)^* \simeq P_1^* \oplus P_2^*$ and $R^* \simeq R$ as a right R-module. Hence, if P is a direct summand of R^n then so is P^*.

3. (12 points). Factor the ideals (3) and (11) into the product of prime ideals in the ring of integers of $\mathbb{Q}(\sqrt{5})$.

Solution: The ring of integers in $\mathbb{Q}(\sqrt{5})$ is $\mathbb{Z}[a]$, where $a = (\sqrt{5}-1)/2$, so a is a root of $t^2 + t - 1 = 0$. Modulo 3 this polynomial remains irreducible, so (3) remains prime. Modulo 11 it factors as $(t-3)(t+4)$, hence $(11) = (11, a-3) \cdot (11, a+4)$.

4. (12 points). Let G be a nonabelian group of order p^3, where p is a prime. What are the dimensions of irreducible complex representations of G?

Solution: since G is nonabelian, the commutant G' is of order p. Therefore, G has p^2 one-dimensional representations. Since the dimensions of irreducible representations divide $|G|$ and their squares sum up to $|G| = p^3$, the remaining irreducible representations have dimension p (and there are $p-1$ of them).

5. (12 points). Let A be a semisimple finite-dimensional $\mathbb{C}$-algebra, M and N finite-dimensional A-modules. Prove that $\dim_k \mathrm{Hom}_A(M, N) = \dim_k \mathrm{Hom}_A(N, M)$.

Solution: Let $V_1, \dots, V_n$ be all simple A-modules. Then $M = \oplus V_i^{m_i}$ and $N = \oplus V_i^{n_i}$ (since A is semisimple). Now both dimensions are equal to $\sum_i m_i n_i$ (by Schur's lemma).