Qualifying Exam in Algebra, Fall 2007

Part I. Definitions and Theorems.

1 (6 points). What is an adjoint pair of functors?
2 (6 points). State a theorem about the structure of the Jacobson radical of an artinian ring.

Part II. True or false. Give brief justification.

1 (8 points). For any nontrivial finite group G and field k the group ring $k[G]$ has zero divisors.
2 (8 points). If E/K is a normal extension then there exist an extension E'/E such that E'/K is Galois.
3 (8 points). There exists a finite group G with precisely four inequivalent irreducible representations of dimension 1,2,3 and 4.
4 (8 points). There exists a ring R such that $[R] \in K_0(R)$ is divisible by 2.
5 (8 points). Any ideal in $\mathbb{C}[x, y]$ is generated by two elements.

Part III. Longer problems.

You have to solve any 4 of the problems below.

1 (12 points). Compute the number of nonzero proper left ideals in $Mat_2(\mathbb{F}_5)$.
2 (12 points). Let E/K be a Galois extension of degree 90. Prove that there exists an intermediate field $E \supset F \supset K$ such that $|F/K| = 6$.
3 (12 points). Let R be a ring and $x = x^2 \in J(R)$. Prove that $x = 0$.
4 (12 points). How many irreducible factors does $x^{255} - 1 \in \mathbb{Q}[x]$ have and what are their degrees?
5 (12 points). Find the associated primes of the ideal $(x^2, xy) \subset \mathbb{C}[x, y]$.

Qualifying Exam in Algebra, Fall 2007 Solutions

Part I. Definitions and Theorems.

1 (6 points). What is an adjoint pair of functors?

Answer: An adjoint pair of functors is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and bijections $\mathrm{Hom}(F(C), D) = \mathrm{Hom}(C, G(D))$ for any $C \in Ob(\mathcal{C}), D \in Ob(\mathcal{D})$ that are natural in C and D.

2 (6 points). State a theorem about the structure of the Jacobson radical of an artinian ring.

Answer: Jacobson radical of an artinian ring is nilpotent.

Part II. True or false. Give brief justification.

1 (8 points). For any nontrivial finite group G and field k the group ring $k[G]$ has zero divisors.

Solution. True: consider $x = \sum_g \in Gg \in k[G]$, then for any $h \in G$ $(h-1)x = 0$.

2 (8 points). If E/K is a normal extension then there exist an extension E'/E such that E'/K is Galois.

Solution. False: if E/K is not separable then it is not contained in a Galois extension.

3 (8 points). There exists a finite group G with precisely four inequivalent irreducible representations of dimension 1,2,3 and 4.

Solution. False: $|G| = 1^2 + 2^2 + 3^2 + 4^2 = 30$ but 30 is not divisible by 4.

4 (8 points). There exists a ring R such that $[R] \in K_0(R)$ is divisible by 2.

Solution. True: $R = Mat_2(\mathbb{C})$ is an example.

5 (8 points). Any ideal in $\mathbb{C}[x, y]$ is generated by two elements.

Solution. False: ideal $I = (x^2, xy, y^2)$ is not generated by 2 elements. Indeed assume $I = (f, g)$. Then the polynomials f, g have no linear and constant terms, so can be written as $f = f_2 +$ terms of degree ≥ 3 and $g = g_2 +$ terms of degree ≥ 3. The polinomials x^2, xy, y^2 should be expressible as linear combinations of f_2 and g_2 with constant coefficients which is impossible since x^2, xy, y^2 are linearly independent.

Part III. Longer problems.

You have to solve any 4 of the problems below.

1 (12 points). Compute the number of nonzero proper left ideals in $Mat_2(\mathbb{F}_5)$.

Solution. The ring $R = Mat_2(\mathbb{F}_5)$ has exactly one simple module L of dimension 2 over $\mathbb{F}_5$ and clearly any nonzero proper ideal of R is isomorphic to L (since $R = L \oplus L$ as R–module). Now $\dim(Hom(L,R)) = 2$ and for $\phi, \psi \in Hom(L,R)$ the condition $Im(\phi) = Im(\psi)$ is equivalent to proportionality of ϕ and ψ (since $End_R(L) = \mathbb{F}_5$). Thus we need to count number of nonzero vectors in 2-dimensional vector space over $\mathbb{F}_5$ up to proportionality, which is 6.

Answer: 6.

2 (12 points). Let E/K be a Galois extension of degree 90. Prove that there exists an intermediate field $E \supset F \supset K$ such that $|F/K| = 6$.

Solution. Let $G = Gal(E/K)$. Then $|G| = 90$ and we need to prove that G contains a subgroup of index 6. Number of Sylow 5-subgroups P of G is 1 or 6; in either case $|N(P)|$ is divisible by 3. Let $Q \subset N(P)/P$ be a subgroup of order 3 (it exists by Sylow theory); then its preimage in $N(P)$ is of order 15 and hence of index 6 in G.

3 (12 points). Let R be a ring and $x = x^2 \in J(R)$. Prove that $x = 0$.

Solution. Assume that $x \neq 0$. Consider the left ideal $R(x-1)$. It is nontrivial since $1 = r(x-1)$ implies $x = r(x^2 - x) = 0$. Let M be a maximal left ideal that contains $R(x-1)$ (it exists by Zorn's Lemma). Then $x \notin M$ since otherwise $1 = x - (x-1) \in M$. Hence $x \notin J(R)$ since $J(R)$ is an intersection of maximal left ideals. Contradiction.

4 (12 points). How many irreducible factors does $x^{255} - 1 \in \mathbb{Q}[x]$ have and what are their degrees?

Solution. The irreducible factors are cyclotomic polynomials $\Phi_m(x)$ where m is a divisor of 255. Since $255 = 3 \cdot 5 \cdot 17$ we have 8 factors of degrees $1, 2, 4, 16, 8, 32, 64, 128$ (degree of $\Phi_m(x)$ is $\phi(m)$).

Answer: 8 factors of degree $1, 2, 4, 8, 16, 32, 64, 128$.

5 (12 points). Find the associated primes of the ideal $I = (x^2, xy) \subset \mathbb{C}[x,y]$.

Solution. The ideal I is not primary since $xy \in I$ but $y^m \notin I$. We claim that $I = (x) \cap (y, x^2)$ is an irredundant primary decomposition. Then the associated primes are $\sqrt{(x)} = (x)$ and $\sqrt{(y, x^2)} = (x, y)$.

Answer: two primes (x) and (x, y).

1 (12 points). Compute the number of nonzero proper left ideals in $Mat_2(\mathbb{F}_5)$.

Solution. The ring $R = Mat_2(\mathbb{F}_5)$ has exactly one simple module L of dimension 2 over $\mathbb{F}_5$ and clearly any nonzero proper ideal of R is isomorphic to L (since $R = L \oplus L$ as R–module). Now $\dim(Hom(L,R)) = 2$ and for $\phi, \psi \in Hom(L,R)$ the condition $Im(\phi) = Im(\psi)$ is equivalent to proportionality of ϕ and ψ (since $End_R(L) = \mathbb{F}_5$). Thus we need to count number of nonzero vectors in 2-dimensional vector space over $\mathbb{F}_5$ up to proportionality, which is 6.

Answer: 6.

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Answer: two primes (x) and (x, y).