ALGEBRA QUALIFYING EXAM

Fall 2006

I. Definitions and Theorems

1. (a) Define Galois extensions.
   (b) State the Fundamental theorem of Galois theory with as many details as possible.
2. (a) Define equivalence of categories.
   (b) State a theorem characterizing rings whose categories of left modules are equivalent.
3. Give three different definitions of a projective module.

II. True or False questions. Give brief but to the point justification.

1. If $k \subset F$ and $F \subset L$ are normal field extensions, then the extension $k \subset L$ is also normal.
2. The dihedral group D_9 has a three-dimensional irreducible representation over $\mathbb{C}$.
3. Every finitely generated commutative algebra is Noetherian.
4. Morita equivalent rings have isomorphic lattices of left ideals.
5. Every finitely generated flat module over a PID is free.
6. Two affine varieties (X, A) and (Y, B) over $\mathbb{C}$ are isomorphic if and only if the algebras A and B are isomorphic.

III. Longer problems. Do any *four* of the following problems.

1. Let G be a group of order p^4, where p is prime, such that the center $Z(G)$ of G has order p^2. Find the number of conjugacy classes of G.
2. Show that the functor $F : \mathsf{Rings} \to \mathsf{Groups}$ such that $F(R) = R^*$ (the group of units of R) has a left adjoint. Does it have a right adjoint?
3. Classify up to similarity all linear transformations $T \in \mathrm{End}(\mathbb{C}^6)$ such that T has exactly one two-dimensional invariant subspace.
4. Let V be a finite dimensional vector space and let $R = \Lambda^*(V)$ be its exterior algebra. Find the Jacobson radical of R.
5. Let $R \subset S$ be an integral extension of rings where S is an integral domain. Prove that R is a field if and only if S is a field.