Part I. Definitions/Theorems.
1. Give three different definitions of a finite nilpotent group.
2. Define the transcendence degree of a field extension $E \subseteq F$, making sure you include the definitions of all related language that you need.
3. What is a natural transformation $\eta : F \to G$ between two functors?

Part II. True/False. Justify your answers briefly.
1. All groups of order 175 are abelian.
2. If $R$ is a ring with no non-trivial left ideals, then it also has no non-trivial right ideals.
3. If $M$ is a projective right $R$-module, then the functor $M \otimes_R ? : R$-mod $\rightarrow R$-mod is exact.
4. The smallest $n$ such that the alternating group $A_n$ has a subgroup of order 15 is $n = 8$.
5. Let $f, g : V \to V$ be endomorphisms of a finite dimensional vector space $V$ and let $f \otimes g : V \otimes_F V \to V \otimes_F V$ be their tensor product. Then, the minimal polynomial $m_{f \otimes g}(x)$ of $f \otimes g$ is the product $m_f(x)m_g(x)$ of the minimal polynomials of $f$ and $g$.
6. Let $R$ be an integral domain with field of fractions $F$. Then, $r \in R$ is a unit if and only if $\frac{1}{r} \in F$ is integral over $R$.

Part III. Longer problems. Attempt any FOUR of the following five questions.
1. Let $A$ be the abelian group $\mathbb{Z}^3$ and let $B$ be the subgroup generated by $(1, 2, 1), (1, 1, 3)$ and $(1, 3, 5)$. Compute the order of the quotient group $A/B$, explaining your answer carefully.
2. Describe the conjugacy classes of the generalized quaternion group $Q_4 = \langle a, b | a^8 = 1, a^4 = b^2, bab^{-1} = a^{-1} \rangle$ of order 16. Hence compute its character table.
3. Suppose that $M$ is an $R$-module with the property that every $R$-submodule of $M$ has a complement. Prove that $M$ is equal to the sum of all its simple $R$-submodules.
4. Let $A$ and $B$ be finite dimensional algebras over an algebraically closed field $F$, and let $A \otimes B$ be their tensor product over $F$, with the usual multiplication defined so that $(a \otimes b)(a' \otimes b') = (aa') \otimes (bb')$ for $a, a' \in A$ and $b, b' \in B$. Prove that the Jacobson radical $J(A \otimes B)$ is equal to $J(A) \otimes B + A \otimes J(B)$.
5. Let $I$ and $J$ be ideals of $A = \mathbb{C}[x, y]$ such that $V(I) \cap V(J) = \emptyset$. Prove that $A/(I \cap J) \cong A/I \times A/J$. 
1. Give three different definitions of a finitely generated group.

(a) Set \( G^0 = G, \quad G^{i+1} = \langle G, G^i \rangle \), subgroup generated by \( x\gamma^{-1}y \gamma^{-1} \neq x\gamma, y\gamma \in G \)

Then \( G \) is nilpotent if \( G^i = 1 \) for \( i \to \infty \).

(b) \( G \) finite is nilpotent if it the direct product of its Sylow p-subgroups.

(c) \( G \) is nilpotent if \( Z(G) \neq 1 \) and \( G/Z(G) \) is nilpotent.

2. Define transcendence degree...

Say \( E/F \) is a field extension, and \( x_i \) (\( i \in I \)) lie in \( F \).

They are algebraically dependent if \( f(x_1, \ldots, x_n) = 0 \) for some \( n \geq 1 \)

and some \( C + f(x_1, \ldots, x_n) \in E[x_1, \ldots, x_n] \).

Otherwise they are algebraically independent.

A transcendence base for \( E/F \) is a maximal algebraic subset of \( E \).

They exist, any two have same cardinality.

\( \text{tr.deg}_F(E) = \) #elts in a transcendence base.

3. What is a radical transformation \( y: F \to G \)?

Let \( F, G: A \to B \) be functors. \( y: F \to G \) means:

\( \forall \gamma \in \text{Ob}(A) \), a map \( \eta_A: FA \to GA \) in \( B \) s.t.

\( y: A \to B \in \text{Arr}(A) \), the diagram commutes.

\[ \begin{array}{ccc}
FA & \xrightarrow{Ff} & FB \\
\downarrow \eta_A & & \downarrow \eta_B \\
GA & \xrightarrow{Gf} & GB
\end{array} \]
1. All groups of order 175 are abelian.
   - 175 = 5\times 35 = 5 \times 5 \times 7
   - \text{by Sylow theorems}
   - its product of Syl_5 and Syl_7 is abelian.

2. R non-trivial left ideal \implies no non-trivial right ideal: dual argument.
   - If \ R is simple, so by Wedderburn, \ R \cong Mn(D) \times \cdots \times Mn(D)
   - In fact \ n=1 as \ R \ simple not just semisimple, so \ R \cong Mn(D)
   - In fact \ n=1 the each column gives a left ideal, so \ R \cong D
   - So every non-zero elt is a unit, so no non-trivial right ideal.

3. \ M_k \text{ projective} \implies \ M_k \otimes \_ R \text{ is exact} \quad \mathbb{Q}
   - \ M_k \text{ is a free module } F_k \text{, so } F_k \otimes \_ R \text{ is exact}.
   - That follow we \ R_k \otimes \_ R \implies Id \text{ is exact}.

4. Smallest \ n \ s.t. \ A_n \ has subgroup of order 15 \implies n=8.

5. For group of order 15, \ m_{xy} \ 5 \times 5 \times 7
   - \ m_{xy} \times 5 \times 5 \times 7

6. R integral domain, F field algebraic \implies x \in R is unit if \ \frac{1}{x} \in F \text{ is integral over } R
   - \ \frac{1}{x} \text{ is not a } x \in R \\implies x \in R [x] \text{ is unit}.
   - \ Say \ \frac{1}{x} \text{ is not a } x^n + a_1 x^{n-1} + \cdots + a_n \in R [x].

   Then \ \frac{1}{x} + a_1 + a_2 x + \cdots + a_n x^{n-1} = 0 \implies \frac{1}{x} \text{ is unit } R \text{ is unit}.
1. \( H = \mathbb{Z}^3 \), \( B = \langle (1,2,1), (2,1,3), (1,3,5) \rangle \). Compute \( |A/\vartheta| \).

In subgroup a free group, so free, and exact basis \( x_1, x_2, x_3 \) for \( A \) and \( d_1, d_2, d_3 \) s.t. \( d_1 x_1, d_2 x_2, d_3 x_3 \) is basis for \( B \). (theorem from class)

These are equivalent to give bases \( A, B \) by unimodular matrix in \( GL_3(\mathbb{Z}) \), so \( d_1, d_2, d_3 \) are elementary divisors \( \det \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 5 \end{pmatrix} \).

\[ A/\vartheta \cong \mathbb{Z}/(d_1) \oplus \mathbb{Z}/(d_2) \oplus \mathbb{Z}/(d_3) \] a order \( d_1d_2d_3 \) \& det \( \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 5 \end{pmatrix} \)

Now compute \[ \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -3 \\ -4 & -6 \\ 0 & 0 \end{pmatrix} \]

\[ \det = -1 \cdot (-1) = -13 \]

Thus \( |A/\vartheta| = 6 \cdot 13 \).

2. Define new group \( G_4 = \langle a, b | a^4 = b^2, b^{-1}a^4b = a \rangle \) and find it character table.

Element are \( 1, a, a^2, a^3, a^4, a^5, a^6, a^7 \), some normal subgroups: \( \langle a^2 \rangle, \langle a^3 \rangle, \langle a^4 \rangle \)

\( G_4 \) 

\( \begin{array}{c|cccccccccc} \text{Class} & 1 & a & a^2 & a^3 & a^4 & a^5 & a^6 & a^7 \\ \hline \text{Size} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \text{Triv.} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & \text{even} & \text{even} & \text{even} & \text{even} & \text{even} & \text{even} & \text{even} & \text{even} \\ \text{1st lift} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \end{pmatrix} \\ \text{2nd lift} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \end{pmatrix} \\ \text{3rd lift} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \end{pmatrix} \\ \end{array} \)

\( a \) is characteristic of \( G \) and \( D_4 \)

\[ \mathbf{a} \text{ is lifted from } D_4 \]

\[ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \end{pmatrix} \]

2. Compute char. table of \( D_4 \) first.
3. If every $R$-submodule of $M$ has a complement, prove $M$ is simple or $R$ is simple.

Let $N \subseteq S$. By assumption, $M = N \oplus K$. For $K \neq 0$, $K$ has no simple submodules.

Let $L \leq K$ be a non-zero cyclic submodule of $K$.

$L$ is quotient of $R$, $R$ has every ideal of $R$ lying in a maximal ideal.

$L$ has a submodule $P$ s.t. $L/P$ is simple module.

$N \oplus P$ has a complement, $Q$, say.

The unique project of the cyclic vector in $L$ to $Q$ generates a module isomorphic to $L/P$.

4. Let $A, B$ be $F$-algebras. Prove $J(A \otimes B) = J(A) \otimes B + A \otimes J(B)$.

Note $A \otimes B$ is f.d., hence Artinian. So sufficient to show:

1. $J(A) \otimes B + A \otimes J(B)$ is nilpotent ideal.
2. $A \otimes B/(J(A) \otimes B + A \otimes J(B))$ is semisimple module.

Consider $x \otimes a + a \otimes y$ (general)

$J(A) \otimes B + A \otimes J(B)$ gen.

For $i = 1, \ldots, n$,

$x_{i1}, a_{i1} x_{i2}, \ldots \otimes b_{i1} y_{i2} \ldots$

Each term has $\geq \frac{1}{2}$ $x_i$ or two have $\geq \frac{1}{2}$ $y_i$.

Since $J(A)$ and $J(B)$ are nilpotent, each $x_i$ is always zero for $n \to \infty$.

Hence, $\max F$ of $A \otimes B$.

And if $M$ is a simple $A$-module, $N$ is a simple $B$-module, then $M \otimes N$ is a simple $A \otimes B$-module.
5. \( I, J \subseteq A = \mathbb{C}[x_{ij}], \quad V(I) \cap V(J) = \emptyset \)

Show \( A/_{\text{Inj}} \cong A/I \times A/J \).

Define map \( A \to A/I \times A/J, \quad a \mapsto (a+I, a+J) \)

kered in \( \text{Inj} \), so \( A/_{\text{Inj}} \cong A/I \times A/J \).

For surjectivity, \( V(I) \cap V(J) = V(I+J) = \emptyset \)

\( I+J \in \mathbb{R} \) by Nullstellensatz

\( I = i+j \) for \( i \in I, j \in J \).

Take any \( a \in A, b \in A \). Then \( b + a \mapsto (a+I, b+J) \).