Algebra Qualifying Exam

Fall 2004

Conventions: Throughout this examination, assume all rings have identities and all modules are unitary.

Part I Theorems. Carefully state each of the following;

1. The Sylow theorems.
2. The Hilbert Nullstellensatz (in any form).
3. At least three equivalent deifnitions of projective modules.

Part II True–False. Determine whether each statement is true or false. Support your decision by a brief explanation or a counterexample.

1. There are no simple groups of order 80.
2. If R is a PID then the polynomial ring $R[x]$ is also a PID.
3. If R is a commutative ring without zero divisors then any submodule of a free R-module is projective.
4. Let $\mathbb{Z}_2$ be a $\mathbb{Z}_6$-module via the projection $\mathbb{Z}_6 \to \mathbb{Z}_6/2\mathbb{Z}_6 = \mathbb{Z}_2$. Then $\mathrm{Ext}_{\mathbb{Z}_6}(\mathbb{Z}_2, \mathbb{Z}_6) = 0$.
5. If D is a division ring and M is a nonzero left D-module then $\mathrm{End}(M)$ is a simple ring.
6. If R is a PID and $a \in R \setminus \{0\}$ then the ring $R/(a)$ is Noetherian and Artinian.
7. If u and v are algebraic over a field K then so is $u + v$.

Part III Problems. Give complete solutions for each of the following.

1. Determine the Galois group of $X^3 - 5$ over $\mathbb{Q}$, describe explicitly all the subfields of its splitting field, and decide which of these subfields are normal over $\mathbb{Q}$.
2. If N is a non-trivial normal subgroup of a finite nilpotent group G then $N \cap C(G) \neq \{e\}$ where $C(G)$ is the center of G.
3. If M is a Noetherian (left) module over a ring then every epimorphism $f : M \to M$ is an isomorphism.
4. $\mathbb{C}^7$ is not a finite union of its proper subvarieties.

Solutions

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Part I Theorems. Carefully state each of the following;

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Part II True–False. Determine whether each statement is true or false. Support your decision by a brief explanation or a counterexample.

1. There are no simple groups of order 80.
True. Let G be a group of order 80. If its 5-Sylow subgroup is unique it is normal. If not then the Sylow theorems give 16 5-Sylow subgroups that cover 64 elements of order 5. This leaves room for just one 2-Sylow subgroup which is normal.
2. If R is a PID then the polynomial ring $R[x]$ is also a PID.
False. If $R = \mathbb{R}[y]$ then $R[x] = \mathbb{R}[y, x]$ is not PID.
3. If R is a commutative ring without zero divisors then any submodule of a free R-module is projective.
False. Take $R = \mathbb{R}[x, y]$ as a module over itself and its submodule $M = (x, y)$. It is not projective since the natural projection $R \oplus R \to M$ does not split.
4. Let $\mathbb{Z}_2$ be a $\mathbb{Z}_6$-module via the projection $\mathbb{Z}_6 \to \mathbb{Z}_6/2\mathbb{Z}_6 = \mathbb{Z}_2$. Then $\mathrm{Ext}_{\mathbb{Z}_6}(\mathbb{Z}_2, \mathbb{Z}_6) = 0$.
True. $\mathbb{Z}_2$ is a direct summand of the free module $\mathbb{Z}_6$ whence projective and every extension of it splits.
5. If D is a division ring and M is a nonzero left D-module then $\mathrm{End}(M)$ is a simple ring.
False. If M is not finite dimensional then $\mathrm{End}(M)$ has the proper ideal consisting of all the endomorphisms of finite ranks.
6. If R is a PID and $a \in R \setminus \{0\}$ then the ring $R/(a)$ is Noetherian and Artinian.
True. The ideals of $R/(a)$ correspond to the ideals of R containing a whence generated by factors of a. Hence they form a finite set.
7. If u and v are algebraic over a field K then so is $u + v$.
True. The fields $K(u)$ and $K(v)$ are finite-dimensional over k, whence so is $K(u)(v)$. The latter field contains $u + v$ implying $u + v$ is algebraic over K.

Part III Problems. Give complete solutions for each of the following.

1. Determine the Galois group of $X^3 - 5$ over $\mathbb{Q}$, describe explicitly all the subfields of its splitting field, and decide which of these subfields are normal over $\mathbb{Q}$.
The discriminant is $D = -27(-5)^2 = -3^3 25$ which is not a square in $\mathbb{Q}$. Thus the Galois group is S_3. The roots of the polynomial are $\alpha, \alpha\omega, \alpha\bar{\omega}$ where $\alpha = \sqrt[3]{5} \in \mathbb{R}$ and $\omega = 1/2(-1 + i\sqrt{3})$. Thus the splitting field is $\mathbb{Q}(i\sqrt{3}, \sqrt[3]{5})$.

The subfield corresponding to $A_3 \subset S_3$ is $\mathbb{Q}(\sqrt{D}) = \mathbb{Q}(i\sqrt{3})$ and this is the only normal extension. There are 3 other extensions corresponding to subgroups generated by transpositions permuting two of the three roots. The first extension is $\mathbb{Q}(\alpha)$ (the respective transposition is the complex conjugation), the second one is $\mathbb{Q}(\alpha\omega)$, and the third one is $\mathbb{Q}(\alpha\bar{\omega})$.

2. If N is a non-trivial normal subgroup of a finite nilpotent group G then $N \cap C(G) \neq \{e\}$ where $C(G)$ is the center of G.

Since N is nilpotent it is the direct sum of its Sylow subgroups as well as G. Let p be a prime dividing the order of N, N' the p-Sylow subgroup of N, and G' the p-Sylow subgroup of G. Clearly N' is a normal subgroup of G'.

Now consider the action of G' on N' by conjugation. Every orbit has cardinality either 1 or divisible by p and the disjoint union of the orbits is N' whence its cardinality is divisible by p. Since the cardinality of the orbit of e is 1 there is another element $n \in N'$ whose orbit has also the cardinality 1. This implies that $n \in C(G')$ and since G' is a direct summand of G we have $n \in N \cap C(G)$.

3. If M is a Noetherian (left) module over a ring then every epimorphism $f : M \to M$ is an isomorphism.

Suppose $\mathrm{Ker} f \neq 0$. Put $M_n = \mathrm{Ker} f^n$ for every natural n. These submodules of M form a nondecreasing sequence. Moreover it is easy to see by induction that the sequence is really increasing. Indeed the above supposition is the base of the induction and if $x \in M_{n-1} \setminus M_{n-2}$ then $f^{-1}(x) \subset M_n \setminus M_{n-1}$. This contradicts the condition that M is Noetherian.

4. $\mathbb{C}^7$ is not a finite union of its proper subvarieties.

The ideal J of the subvariety $\mathbb{C}^7$ of $\mathbb{C}^7$ is 0. Thus J is prime (since the polynomial ring over $\mathbb{C}$ is an integral domain). This is equivalent to the irreducibility of the variety.