ALGEBRA QUALIFYING EXAM, FALL 2003

Conventions: All rings have identity, and all modules are unital.

Part I. Carefully state each of the following (be certain that any *special* terminology or notation is explained):

1. Hilbert Nulstellensatz.
2. Sylow Theorems.
3. Jordan-Hölder Theorem for modules.

Part II. For each of the following determine whether it is true or false. If true, give a brief explanation. If false, provide a counterexample:

1. If G is a group such that any finitely generated subgroup of G is cyclic, then G is cyclic.
2. If G is a finite nilpotent group, and m is a positive integer dividing $|G|$, then there exists a subgroup of G of order m.
3. If F is an infinite field, then its multiplicative group $F^{\times}$ is never cyclic.
4. Let R be a noetherian local ring with maximal ideal M and let $x_1, \ldots, x_n \in M$. If $\{x_1 + M^2, \ldots, x_n + M^2\}$ is a basis of the R/M-vector space M/M^2 then $M = Rx_1 + \cdots + Rx_n$.
5. If R is a noetherian commutative ring, then every $R/J(R)$-module is semisimple.
6. Let F be an algebraically closed field of characteristic $p > 0$, and $f : F \to F, a \mapsto a^p$ be the Frobenius homomorphism. True or false:
   (a) f is a homeomorphism in the Zariski topology.
   (b) f is an isomorphism of algebraic sets.

Part III. Give complete solutions for each of the following problems.

1. Let G be a finite group. We choose an element $g \in G$ randomly. Then replace it and make another random choice of an element $h \in G$. Prove that the probability that g and h commute equals to $k/|G|$, where k is the number of conjugacy classes in G.
2. Let p be a prime. Then there are exactly $(q^p - q)/p$ monic irreducible polynomials of degree p in $\mathbb{F}_q[x]$ (q is not necessarily a power of p).
3. Let G be a finite group and F be an algebraically closed field of characteristic $p \geq 0$. Prove the following:
   (i) Up to isomorphism, that there are only finitely many irreducible FG-modules $L_1, \ldots, L_k$.

(ii) Let $d_i = \dim L_i$, $1 \leq i \leq k$. Then $\sum_{i=1}^{k} d_i^2 \leq |G|$, and the equality holds if and only if $p \not| |G|$.

(iii) Is it true that the inequality $\sum_{i=1}^{k} d_i^2 \leq |G|$ holds even if F is not algebraically closed?

4. Let R be a ring and V_1, V_2 be non-isomorphic simple R-modules. Prove that $V_1 \oplus V_2$ has exactly four submodules: 0, $0 \oplus V_2$, $V_1 \oplus 0$, and $V_1 \oplus V_2$.
5. Let V and W be vector spaces over the same field F of finite dimensions n and m, respectively. Assume that $f : V \to V$ and $g : W \to W$ are linear transformations. Prove that $\det f \otimes g = (\det f)^m (\det g)^n$.