Algebra Qualifying Exam

Fall 2002

Conventions: Throughout this examination, assume all rings have identities and all modules are unitary.

Part I Theorems. Carefully state each of the following theorems:

1. The fundamental theorem of Galois theory (in as complete form as you can).
2. The Hilbert Nullstellensatz (in any form).
3. The Wdderburn-Artin theorem for simple rings.

Part II True–False. Determine whether each statement is true or false. If true, give a brief explanation. If false, provide a counterexample.

1. Every finite group of an even order has a subgroup of index 2.
2. If R is a commutative ring, I is an ideal of R, and S is a multiplicative subset of R, then $\sqrt{S^{-1}I} = S^{-1}(\sqrt{I})$.
3. If a ring R is commutative then any submodule of a projective R-module is projective.
4. An affine algebraic variety cannot have an infinite strictly decreasing sequence of (algebraic) subvarieties.
5. If M_1 and M_2 are respectively right and left modules over a division ring D such that $M_1 \otimes_D M_2 = 0$ then either $M_1 = 0$ or $M_2 = 0$.
6. The ring $\mathbb{R}[x]$ is semisimple.
7. For every finite field F the multiplicative group F^* is cyclic.

Part III Problems. Give complete solutions for each of the following.

1. Determine the Galois group of $X^3 + 11$ over $\mathbb{Q}$, determine all subfields of its splitting field, and decide which of these subfields are normal over $\mathbb{Q}$. Exhibit generators for at least two of the subfields.
2. If N is a non-trivial normal subgroup of a p-group G (p is a prime integer) then $N \cap C(G) \neq \{e\}$ where $C(G)$ is the center of G.
3. Prove that no group of order 48 is simple.
4. Find all the singular points of the curve $y^2 = x(x-1)^2$ in $\mathbb{C}^2$.
5. Let n be a positive integer and R the ring of all $n \times n$ lower triangular matrices over a division ring D (non-zero diagonal entries are allowed). Find the Jacobson radical $J(R)$ and describe the factorization of $R/J(R)$ according to the Weddeburn-Artin theorem.

Solutions

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Part II True–False. Determine whether each statement is true or false. If true, give a brief explanation. If false, provide a counterexample.

1. Every finite group of an even order has a subgroup of index 2.
False. A_3 having order 12 does not have a subgroup of order 6.
2. If R is a commutative ring, I is an ideal of R, and S is a multiplicative subset of R, then $\sqrt{S^{-1}I} = S^{-1}(\sqrt{I})$.
True. If $a^n = (x/s)^n \in S^{-1}I$ then $s_1x^n \in I$ for some $s_1 \in S$ whence $s_1x \in \sqrt{I}$ and $a = s_1x/s_1s \in S^{-1}\sqrt{I}$. Conversely if $x^n \in I$ then $(x/s)^n \in S^{-1}I$ for every $s \in S$.
3. If a ring R is commutative then any submodule of a projective R-module is projective.
False. Take $R = \mathbb{Z}_4$ and $M = 2R$. Since $2M = 0$ it is not projective.
4. An affine algebraic variety cannot have an infinite strictly decreasing sequence of (algebraic) subvarieties.
True. The ring $F[X]$ is Noetherian.
5. If M_1 and M_2 are respectively right and left modules over a division ring D such that $M_1 \otimes_D M_2 = 0$ then either $M_1 = 0$ or $M_2 = 0$.
True. If B_i is a basis of M_i then $\{b \otimes b' | b \in B_1, b' \in B_2\}$ is a basis of $M_1 \otimes M_2$.
6. The ring $\mathbb{R}[x]$ is semisimple.
True. For every $a \in \mathbb{R}$ the ideal $M_a = R(x-a)$ is maximal and $\cap_a M_a = 0$ whence $J(R) = 0$.
7. For every finite field F the multiplicative group F^* is cyclic.
True. $G = F^*$ is a finite Abelian group whence $G = \oplus_i \mathbb{Z}_{m_i}$. If $m = \prod_i m_i$ then all elements of G are roots of $x^m - 1$ whence there are precisely m of them.

Part III Problems. Give complete solutions for each of the following.

1. Determine the Galois group of $X^3 + 11$ over $\mathbb{Q}$, determine all subfields of its splitting field, and decide which of these subfields are normal over $\mathbb{Q}$. Describe at least two of the subfields by their generators.
The roots are $a_1 = -\sqrt[3]{11}$, $a_2 = a_1\zeta$, and $a_3 = a_1\zeta^2$ where ζ is a primitive third root of 1. Thus the splitting field F is $F = \mathbb{Q}[i\sqrt{3}, \sqrt[3]{11}]$. The discriminant is $D = -27 \times 121$. Since $D \notin \mathbb{Q}^2$ the group is S_3. There are 4 proper subgroups: $A_3, <(12)>, <(13)>, <(23)>$. Denote them G_i in the above order and denote by F_i the respective fixed fields.
The field F_1 is generated by the square root of D, i.e., $F_1 = \mathbb{Q}[i\sqrt{3}]$. Since (23) is the complex conjugation, $F_4 = \mathbb{Q}[\sqrt[3]{11}]$.

2. If N is a non-trivial normal subgroup of a p-group G (p is a prime integer) then $N \cap C(G) \neq \{e\}$ where $C(G)$ is the center of G.
Consider the action of G on N by the conjugation. Each orbit has either cardinality 1 or a positive power of p and their cardinalities add up to a positive power of p. Thus since 1 is its own orbit there is another $n \in N$ with the same property. Hence $n \in C(G)$.
3. Prove that no group of order 48 is simple.
Let $|G| = 48 = 2^4 \times 3$. If G is simple then a Sylow 2-subgroup S is not normal and has index 3. The (left) multiplication in G defines a non-trivial action of G on the set of (left) cosets of S whence a non-trivial homomorphism $G \to S_3$. Since again G is simple it is an embedding which is impossible.
4. Find all the singular points of the curve $y^2 = x(x-1)^2$ in $\mathbb{C}^2$.
Take a point (u,v) on the curve C and shift the coordinates to this point. The equation becomes $(y+v)^2 = (x+u)(x+u-1)^2$. The linear part is $2vy = (u-1)^2x + 2u(u-1)x$ or $2vy = (u-1)(3u-1)x$. This is an equation of the tangent line to C at (u,v). This line becomes the whole plane iff $(u,v) = (1,0)$. Thus this is the only singular point of C.
5. Let n be a positive integer and R the ring of all $n \times n$ lower triangular matrices over a division ring D (non-zero diagonal entries are allowed). Find the Jacobson radical $J(R)$ and describe the factorization of $R/J(R)$ according to the Weddeburn-Artin theorem.
Let I be the set of all strictly lower triangular matrices. It is a two-sided ideal in R. Also every element of it is regular since $1 + M$ is invertible for every $N \in I$. Thus $I \subset J(R)$. Also $R/I = \times D$ (n times) whence semisimple. Thus $I = J(R)$ and the factorization is above.