

MA457, SECOND REVIEW

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The second third of our course was about the Cantor set, symbolic dynamics, and chaos. We've also started just a bit of the last third, where we've been talking about Julia sets and the Mandelbrot set. At this point you should understand—and be able to explain and use—everything in the following summary.

1. THE CANTOR SET

Start with the interval $[0, 1]$ and remove the open middle third $(\frac{1}{3}, \frac{2}{3})$ —this leaves the two intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Remove the open middle third of each of these two pieces, leaving four closed intervals. Then remove the open middle third of each of these four pieces, etc. The Cantor set K consists of all points in $[0, 1]$ which never get removed.

Theorem 1.1. *The Cantor set K has the following properties:*

- (a) K contains the endpoints of all intervals removed during its construction—consequently, K is infinite.
- (b) The length of K is 0.
- (c) Between any two points of K there is a point which is not in K .
- (d) K contains no non-trivial closed intervals.
- (e) K consists of all numbers in $[0, 1]$ which can be represented in base 3 by an expansion consisting only of 0's and 2's. Consequently, K contains many points which are not endpoints of removed intervals.
- (f) K is uncountable.

1.2. Dynamics of the tent map. The Cantor set comes up when we study the dynamics of the

tent map $T: \mathbb{R} \rightarrow \mathbb{R}$ defined by $T(x) = \begin{cases} 3x & \text{if } x < \frac{1}{2} \\ 3 - 3x & \text{if } x \geq \frac{1}{2} \end{cases}$. Note the following observations:

- (1) If $x < 0$ or $x > 1$ then the orbit of x tends towards $-\infty$.
- (2) If $x \in (\frac{1}{3}, \frac{2}{3})$ then $T(x) > 1$, and so the orbit of x tends towards $-\infty$.
- (3) If $x \in (\frac{1}{9}, \frac{2}{9})$ or $x \in (\frac{7}{9}, \frac{8}{9})$ then $T(x) \in (\frac{1}{3}, \frac{2}{3})$ and so the orbit of x goes to $-\infty$.
- (4) The Cantor set K is precisely the set of points in $[0, 1]$ whose orbit never leaves $[0, 1]$. So if $x \notin K$ then the dynamics of T are not very interesting, but if we look only at $T: K \rightarrow K$ then things are more complicated.
- (5) We understand how T behaves on the Cantor set by using base 3 expansions. For sequences of 0's and 2's we have

$$T(0.0a_1a_2\dots) = 0.a_1a_2\dots \quad \text{and} \quad T(0.2a_1a_2\dots) = \tilde{a}_1\tilde{a}_2\dots$$

Using this description we can find cycles for T , as well as non-periodic points.

1.3. The quadratic family $f(x) = x^2 + c$ when $c < -2$. The dynamics here are very similar to the tent map we looked at in the last section. Recall the two fixed points p_- and p_+ .

- (1) If $x < -p_+$ or $x > p_+$ then the orbit of x goes to ∞ . So all the interesting dynamics are in the interval $[-p_+, p_+]$.

- (2) There is a ‘middle interval’ inside $[-p_+, p_+]$ consisting of points x such that $T(x) < -p_+$, and such points have orbits going to ∞ . We throw these out, leaving two remaining intervals. Inside each remaining interval there is a middle interval consisting of points x such that $T^2(x) < -p_+$, and we throw these out leaving four remaining intervals. Keep going. What’s left at the end is a set Λ , consisting of all points x whose orbit never leaves $[-p_+, p_+]$. Λ is very similar to the Cantor set.
- (3) We can describe points $x \in \Lambda$ by sequences of L ’s and R ’s, depending on whether they are to the left or right of each removed middle interval. In class we called this sequence the ‘code’ of x . The dynamics of f are described by the rules

$$f(0.Ls_1s_2s_3\dots) = \tilde{s}_1\tilde{s}_2\tilde{s}_3\dots \quad \text{and} \quad f(0.Rs_1s_2s_3\dots) = 0.s_1s_2s_3\dots$$

Note that this is similar to the tent map, but with L corresponding to 2 and R corresponding to 0 (essentially because the tent map is oriented upside-down from the way the parabola is oriented).

- (4) We can associate another kind of sequence to every point in Λ , called its **itinerary**. This is another—equally useful—way of understanding the dynamics of $f: \Lambda \rightarrow \Lambda$. If $x \in \Lambda$ then the itinerary of x is the sequence $I(x) = 0.s_0s_1s_2\dots$ where s_n is L if $f^n(x)$ is to the left of the first removed interval, and is R if $f^n(x)$ is to the right of the first removed interval. In class we learned that given an $x \in \Lambda$, we know how to go back and forth between its code and its itinerary: given one, we have a procedure for recovering the other.
- (5) If $I(x) = 0.s_0s_1s_2\dots$ then $I(f(x)) = 0.s_1s_2s_3\dots$. So using itineraries converts f into the shift map! This is very useful, since we understand the shift map so well.

2. COORDINATE CHANGES

Suppose we have two dynamical systems $f: M \rightarrow M$ and $g: N \rightarrow N$. In many cases we want to compare these, since maybe they are secretly the same thing in disguise. We say that the two systems are **related by a coordinate change** (or are **conjugate**) if there is a function $\varphi: M \rightarrow N$ such that

- (1) φ is one-to-one and onto, and
 (2) $g \circ \varphi = \varphi \circ f$. That is, the two ways of going around the square

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \varphi \downarrow & & \downarrow \varphi \\ N & \xrightarrow{g} & N \end{array}$$

are the same.

Example 2.1. Let $D: [0, 1) \rightarrow [0, 1)$ be the doubling function. Let Σ be the set of all sequences of 0’s and 1’s, and $\sigma: \Sigma \rightarrow \Sigma$ be the shift map. These are two dynamical systems which are essentially the same. Let $\varphi: [0, 1) \rightarrow \Sigma$ send x to its base 2 expansion. This is not quite a change-of-coordinates in the above sense, because φ isn’t quite well-defined (some numbers have more than one base 2 expansion, e.g. $1 = [1.0] = [0.\bar{1}]$). But it is close enough for our purposes.

Theorem 2.2. *Every quadratic map $f(x) = ax^2 + bx + c$ where $a \neq 0$ can be related by a change-of-coordinates to a map of the form $g(u) = u^2 + d$. In fact, one can choose a linear change of coordinates: $u = \varphi(x) = Px + Q$ for some P, Q with $P \neq 0$.*

This explains, for one thing, why the orbit diagram of the logistic family $f(x) = \lambda x(1 - x)$ is so similar to that of the quadratic family $g(x) = x^2 + c$.

Example 2.3. We also saw in class that $g(u) = u^2 - 2$ is related by a change-of-coordinates to the function $T_2: [0, 1] \rightarrow [0, 1]$ defined by $T_2(x) = \begin{cases} 2x & \text{if } x < \frac{1}{2} \\ 2 - 2x & \text{if } x \geq \frac{1}{2} \end{cases}$. The coordinate change is $u = 2 \cos(\pi x)$. It works because of the trig identity $\cos(2\theta) = 2 \cos(\theta)^2 - 1$.

When two dynamical systems f and g are related by a change of coordinates, everything about them is the same. So if you know cycles of f , you can make cycles for g by applying the coordinate change. On the homework we used this to construct cycles for $g(u) = u^2 - 2$.

3. CHAOS

Let M be a metric space (a set together with a notion of distance $d(x, y)$ between any two points of M), and let $f: M \rightarrow M$ be a function. Devaney says that the dynamical system $f: M \rightarrow M$ is **chaotic** if it satisfies three properties:

- (1) For every $x \in M$ and every $\epsilon > 0$, there is a periodic point p such that $d(x, p) < \epsilon$.
- (2) For every two points $x, y \in M$ and every $\epsilon > 0$, there is a point $z \in M$ and an $n > 0$ such that $d(x, z) < \epsilon$ and $d(y, f^n(z)) < \epsilon$.
- (3) There is a $B > 0$ with the following property: For every $x \in M$ and every $\epsilon > 0$, there is a $y \in M$ and an $n > 0$ such that $d(x, y) < \epsilon$ and $d(f^n(x), f^n(y)) > B$.

Property (1) says that ‘periodic points are dense’. Property (2) is called ‘transitivity’. And property (3) is called ‘sensitivity to initial conditions’.

Example 3.1. (1) The doubling function $D: [0, 1] \rightarrow [0, 1]$ (or equivalently, the shift map $\sigma: \Sigma \rightarrow \Sigma$) is chaotic. You should be able to explain why all the properties hold.

(2) The tent function $T_2: [0, 1] \rightarrow [0, 1]$ is chaotic, because we can convert it to the shift map by using itineraries.

(3) The tent function $T: [0, 1] \rightarrow [0, 1]$ defined by $T(x) = \begin{cases} 3x & \text{if } x < \frac{1}{2} \\ 3 - 3x & \text{if } x \geq \frac{1}{2} \end{cases}$ is chaotic on the

Cantor set (or said differently, $T: K \rightarrow K$ is chaotic). This is because the description of T on base 3 expansions of 0s and 2s is the same as that of T_2 on base 2 expansions. And we already know T_2 is chaotic.

(4) The function $f(x) = x^2 + c$ where $c < -2$ is chaotic on the set Λ . Again, using the code of every point $x \in \Lambda$ we see that f is the same (after a change of coordinates) as the tent map T_2 . Or we could use itineraries, and convert f to the shift map. Either way, we find that it is chaotic.

(5) The function $f(x) = x^2 - 2$ is chaotic on the whole interval $[-2, 2]$, because it is related by a coordinate change to the function $T_2: [0, 1] \rightarrow [0, 1]$ (which we already know is chaotic).

4. JULIA SETS

Definition 4.1. A subset S of \mathbb{C} is called **bounded** if there is a real number R such that $|z| < R$ for all $z \in S$.

Definition 4.2. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a complex function.

(a) The **filled Julia set** of f is the set of all z 's whose orbit is bounded.

(b) The **Julia set** of f is the boundary of the filled Julia set.

We never learned exactly what the ‘boundary’ of a set is, although you may remember the definition from calculus classes. For us it's not that important, since we will always use the *filled* Julia set and never the Julia set itself.

We will restrict our study to the quadratic family of functions $Q_c(z) = z^2 + c$. We let K_c denote the filled Julia set of Q_c . The following are two results we proved in class:

Lemma 4.3. If $|z| > 2$ and $|z| \geq c$, then the orbit of z goes to infinity—i.e., $\lim_{n \rightarrow \infty} Q_c^n(z) = \infty$.

Corollary 4.4. *Suppose $|c| < 2$. If $|Q_c^n(z)| > 2$ for some value of n , then the orbit of z goes to infinity.*

We needed these results because they lead to the following algorithm for computing the Julia set, when $|c| < 2$:

- (1) Pick a point z in \mathbb{C} , and compute the first 100 iterations under the function Q_c .
- (2) If one of these iterates has magnitude larger than 2, then stop—the orbit of z goes to infinity, and z is not in the filled Julia set. We assign the point z a color based on how many iterates it took to get a magnitude bigger than 2: red=very few, yellow=more, blue=even more, etc.
- (3) If none of the first 100 iterates of z had magnitude larger than 2, we will make a guess that the iterates will *never* have magnitude larger than 2—so the orbit of z is bounded, and z is in the filled Julia set. We color the point z black.

The above algorithm is not foolproof: if the first 100 iterates never have magnitude bigger than 2, it might still be true that the 200th iterate will have magnitude bigger than 2. So we never know with absolute certainty if a point really is in the Julia set. Said differently, the algorithm really only gives an *approximation* to the Julia set. By increasing the number of iterations, we get better approximations.

5. THE MANDELBROT SET

Theorem 5.1 (The Fundamental Dichotomy). *For any fixed value of c , exactly one of the following two statements holds. Either*

- (a) *The filled Julia set K_c is connected, or*
- (b) *The filled Julia set has infinitely many pieces, and does not contain any disks inside of it (in fact it is basically a Cantor set).*

Definition 5.2. *The Mandelbrot set \mathcal{M} is defined to be $\{c \mid K_c \text{ is connected}\}$. In words, it is the collection of all c values for which the corresponding filled Julia set is connected.*

Exercise 5.3. Explain why $0 \in \mathcal{M}$.

5.4. Decorations in the Mandelbrot set. In order to understand the Mandelbrot set, we break it up into different pieces. We can first look for all c values such that Q_c has an attractive fixed point. This turns out to be the main cardioid in the Mandelbrot set. Then we will look for c values where Q_c has an attractive two-cycle, or an attractive three-cycle, and so on.

5.5. The main cardioid. To have a fixed point, we need $Q_c(z) = z$. That is, $z^2 + c = z$. Re-write this as $z^2 - z + c = 0$, and the solutions are

$$p_{\pm} = \frac{1 \pm \sqrt{1 - 4c}}{2}.$$

For one of these to be attractive, we need either $|Q'_c(p_-)| < 1$ or $|Q'_c(p_+)| < 1$. Since $Q'_c(z) = 2z$, this says that we want

$$|1 \pm \sqrt{1 - 4c}| < 1$$

(to be clear: we want the inequality to hold for + OR -). Let $w = 1 - 4c$, and write $w = re^{i\theta} = r \cos \theta + (r \sin \theta)i$. Then $\sqrt{w} = \sqrt{r}e^{i\theta/2}$, and our inequality becomes

$$\left(1 \pm \sqrt{r} \cos\left(\frac{\theta}{2}\right)\right)^2 + \left(\sqrt{r} \sin\left(\frac{\theta}{2}\right)\right)^2 < 1.$$

This simplifies to $1 \pm 2\sqrt{r} \cos(\frac{\theta}{2}) + r < 1$, or $\sqrt{r} < \mp 2 \cos(\frac{\theta}{2})$. Finally, squaring both sides gives

$$r < 4 \cos^2\left(\frac{\theta}{2}\right).$$

This is the equation for a cardioid, which you can easily sketch. It intersects the real axis at 0 and 4, with the cusp of the cardioid at 0.

Finally, remember that this cardioid graphs the possibilities for w (since $w = re^{i\theta}$). If we remember that $w = 1 - 4c$, or $c = (1 - w)/4$, then we find that the c values are described by a cardioid which intersects the real axis at $1/4$ and $-3/4$, with the cusp at $1/4$.

5.6. The case of attractive two-cycles. We next look for all c values for which Q_c has an attractive two-cycle. To find the two-cycles, we need to solve $(z^2 + c)^2 + c = z$. This is a degree 4 equation, but we already know two of the solutions—namely, the fixed points p_+ and p_- we already found. Dividing the polynomial $z^4 + 2cz^2 - z + (c^2 + c)$ by the polynomial $z^2 - z + c$ (the latter of which gives the equation for the fixed points), we get $z^2 + z + (c + 1)$. So the points on the two-cycles are the solutions to $z^2 + z + (c + 1) = 0$. These are

$$q_{\pm} = \frac{-1 \pm \sqrt{1 - 4(c + 1)}}{2}.$$

To ensure that this is an *attractive* two-cycle, we need to require that

$$|Q'_c(q_-)| \cdot |Q'_c(q_+)| < 1.$$

This becomes the inequality $|(-1 + \sqrt{1 - 4(c + 1)})(-1 - \sqrt{1 + 4(c + 1)})| < 1$. Multiplying out, we find

$$|4(c + 1)| < 1$$

or

$$|c + 1| < \frac{1}{4}.$$

The c -values satisfying this inequality form a disk with radius $\frac{1}{4}$, centered at -1 .