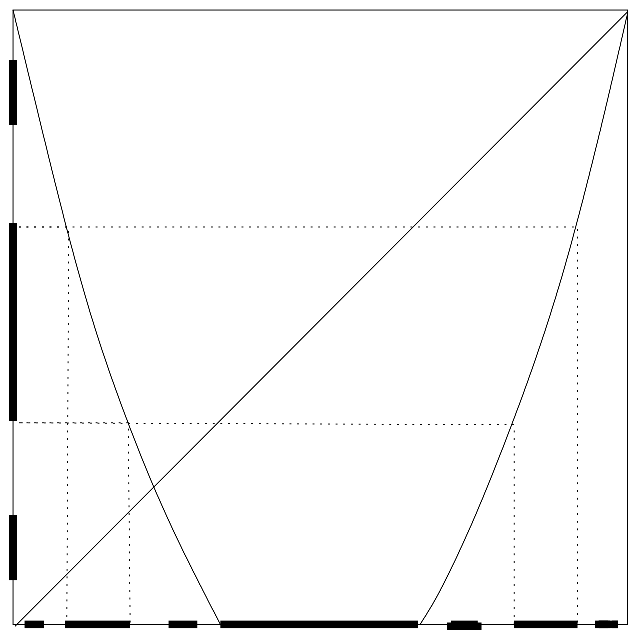
Math 457
Homework Due Wednesday, May 11

1. The diagram below shows a function $f(x) = x^2 + c$, $c < -2$. Let Λ denote all points whose orbit never leaves the given box—this is a generalized Cantor set, and every point $x \in \Lambda$ can be labelled by a code sequence of L's and R's depending on whether x is to the left or right of each removed interval.



   (a) Label a number a on the x-axis whose code would begin $.LRR\ldots$ By using graphical analysis, figure out the first two digits of the code for $f(a)$.

   (b) Do the same for a number b whose code begins $.LLR\ldots$ and a number c whose code begins $.RRL\ldots$ Make sure you give me the first two digits of $f(b)$ and $f(c)$.

   (c) In class we learned exactly how applying f affects the code of a number. Using this knowledge, compute the full orbit of the point p whose code is $.\overline{RL}$. Show the complete graphical analysis of this orbit in your picture, using a different color from what you used above.

2. Let $T(x) = \begin{cases} 2x & \text{if } x < \frac{1}{2} \\ 2-2x & \text{if } x \geq \frac{1}{2}. \end{cases}$

   Recall that in binary $T(0.0a_1a_2a_3\ldots) = 0.a_1a_2a_3\ldots$ and $T(0.1a_1a_2a_3\ldots) = 0.\tilde{a}_1\tilde{a}_2\tilde{a}_3\ldots$.

   Define the *itinerary* of any point $x \in [0,1]$ to be the sequence $I(x) = .s_0s_1s_2s_3\ldots$ where $s_i = L$ if $T^i(x) < \frac{1}{2}$ and $s_i = R$ if $T^i(x) \geq \frac{1}{2}$. Note that $T^0(x) = x$, by convention.

   (a) Determine $I(\frac{1}{7})$.

   (b) Suppose $I(a) = .LRRLL\overline{RL}$. Determine the first nine digits of the binary expansion of a (please remember to use 0's and 1's).

(c) Recall from class that $I(T(x))$ is the shift of $I(x)$. Since $.\overline{LRL}$ is a period 3 point for the shift map, the corresponding point in $[0,1]$ is a period 3-point for T. Determine which point $x \in [0,1]$ has itinerary $.\overline{LRL}$; give x as a rational number.

(d) How many 5-cycles does T have? (Hint: Change this into a problem about the shift map).

(e) (Challenge problem—don't hand in) Find a general rule which allows you to translate from the itinerary of a point x into the binary expansion.

3. On question #2 of the first midterm you met the function $f\colon [0,4] \to [0,4]$ given by

$$f(x) = \begin{cases} 3+x & \text{if } 0 \le x \le 1, \\ -2x+6 & \text{if } 1 \le x \le 3, \\ -3+x & \text{if } 3 \le x \le 4. \end{cases}$$

This function is continuous and *piecewise-linear*, meaning that each piece of it is a linear function.

Your task in this problem is to create a similar function which is piecewise-linear, continuous, and has a 3-cycle. Demonstrate the 3-cycle via graphical analysis. Your function should be of the form $f\colon [0,n] \to [0,n]$ for a whole number n of your choosing.