MA457, THIRD REVIEW

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The last part of our course has been a tour through Julia sets and the Mandelbrot set.

1. Julia sets

Definition 1.1. *A subset S of $\mathbb{C}$ is called* ***bounded*** *if there is a real number R such that $|z| < R$ for all $z \in S$.*

Definition 1.2. *Let $f\colon \mathbb{C} \to \mathbb{C}$ be a complex function.*

(a) The ***filled Julia set*** *of f is the set of all z's whose orbit is bounded.*

(b) The ***Julia set*** *of f is the boundary of the filled Julia set.*

We never learned exactly what the 'boundary' of a set is, although you may remember the definition from calculus classes. For us it's not that important, since we will always use the *filled* Julia set and never the Julia set itself.

We will restrict our study to the quadratic family of functions $Q_c(z) = z^2 + c$. We let K_c denote the filled Julia set of Q_c. The following are two results we proved in class:

Lemma 1.3. *If $|z| > 2$ and $|z| \geq c$, then the orbit of z goes to infinity—i.e., $\lim_{n\to\infty} Q_c^n(z) = \infty$.*

Corollary 1.4. *Suppose $|c| < 2$. If $|Q_c^n(z)| > 2$ for some value of n, then the orbit of z goes to infinity.*

We needed these results because they lead to the following algorithm for computing the Julia set, when $|c| < 2$:

(1) Pick a point z in $\mathbb{C}$, and compute the first 100 iterations under the function Q_c.

(2) If one of these iterates has magnitude larger than 2, then stop—the orbit of z goes to infinity, and z is not in the filled Julia set. We assign the point z a color based on how many iterates it took to get a magnitude bigger than 2: red=very few, yellow=more, blue=even more, etc.

(3) If none of the first 100 iterates of z had magnitude larger than 2, we will make a guess that the iterates will *never* have magnitude larger than 2—so the orbit of z is bounded, and z is in the filled Julia set. We color the point z black.

The above algorithm is not foolproof: if the first 100 iterates never have magnitude bigger than 2, it might still be true that the 200th iterate will have magnitude bigger than 2. So we never know with absolute certainty if a point really is in the Julia set. Said differently, the algorithm really only gives an *approximation* to the Julia set. By increaasing the number of iterations, we get better approximations.

2. The Mandelbrot set

Theorem 2.1 (The Fundamental Dichotomy)**.** *For any fixed value of c, exactly one of the following two statements holds. Either*

(a) The filled Julia set K_c is connected, or

(b) The filled Julia set has infinitely many pieces, and does not contain any disks inside of it (in fact it is basically a Cantor set).

Definition 2.2. *The Mandelbrot set $\mathcal{M}$ is defined to be $\{c \,|\, K_c \text{ is connected}\}$. In words, it is the collection of all c values for which the corresponding filled Julia set is connected.*

How can we use a computer to tell when a filled Julia set is connected or not? If we had to compute the whole Julia set, this would be a pain. An easier way is provided by the next result:

Theorem 2.3. *If the orbit of* 0 *under* Q_c *is bounded, then* K_c *is connected. If the orbit of* 0 *is unbounded, then* Q_c *has infinitely many pieces.*

We didn't *prove* Theorems 2.1 and 2.3 in this course, but you should know their statements. The latter gives us an algorithm for computing the Mandelbrot set:

(1) Pick a c value, and compute the first 100 iterates of 0 under Q_c.
(2) If at some point the orbit of 0 has magnitude greater than 2, then the orbit will go to infinity (by Corollary 1.4). Theorem 2.3 then says that the filled Julia set K_c is not connected, and so this value of c is not in the Mandelbrot set. Assign it a color based on how many iterations it took for the orbit of 0 to have magnitude greater than 2.
(3) If the first 100 iterates of 0 never have magnitude larger than 2, we will guess that this is true for *all* the iterates. In this case the orbit of 0 is bounded, so by Theorem 2.3 the filled Julia set is connected—that is, c is in the Mandelbrot set. We color the point black.

Again, the algorithm is not foolproof—it really only gives us an approximation to the Mandelbrot set. We can improve the approximation by increasing the number of iterations we perform.

3. DECORATIONS IN THE MANDELBROT SET

In order to understand the Mandelbrot set, we break it up into different pieces. We can first look for all c values such that Q_c has an attractive fixed point. This turns out to be the main cardioid in the Mandelbrot set. Then we will look for c values where Q_c has an attractive two-cycle, or an attractive three-cycle, and so on. It is a (hard) theorem that whenever you have an attractive cycle, the orbit of 0 will necessarily converge to it—this is because 0 is the only critical point of Q_c. So the presence of an attractive cycle forces the orbit of 0 to be bounded, and therefore forces c to be in the Mandelbrot set.

3.1. **The main cardioid.** To have a fixed point, we need $Q_c(z) = z$. That is, $z^2 + c = z$. Re-write this as $z^2 - z + c = 0$, and the solutions are

$$p_\pm = \frac{1 \pm \sqrt{1-4c}}{2}.$$

For one of these to be attractive, we need either $|Q_c'(p_-)| < 1$ or $|Q_c'(p_+)| < 1$. Since $Q_c'(z) = 2z$, this says that we want

$$|1 \pm \sqrt{1-4c}| < 1$$

(to be clear: we want the inequality to hold for + OR −). Let $w = 1 - 4c$, and write $w = re^{i\theta} = r\cos\theta + (r\sin\theta)i$. Then $\sqrt{w} = \sqrt{r}e^{i\theta/2}$, and our inequality becomes

$$\Big(1 \pm \sqrt{r}\cos(\tfrac{\theta}{2})\Big)^2 + \Big(\sqrt{r}\sin(\tfrac{\theta}{2})\Big)^2 < 1.$$

This simplifies to $1 \pm 2\sqrt{r}\cos(\frac{\theta}{2}) + r < 1$, or $\sqrt{r} < \mp 2\cos(\frac{\theta}{2})$. Finally, squaring both sides gives

$$r < 4\cos^2(\tfrac{\theta}{2}).$$

This is the equation for a cardioid, which you can easily sketch. It intersects the real axis at 0 and 4, with the cusp of the cardioid at 0.

Finally, remember that this cardioid graphs the possibilities for w (since $w = re^{i\theta}$). If we remember that $w = 1 - 4c$, or $c = (1-w)/4$, then we find that the c values are described by a cardioid which intersects the real axis at $1/4$ and $-3/4$, with the cusp at $1/4$.

3.2. **The case of attractive two-cycles.** We next look for all c values for which Q_c has an attractive two-cycle. To find the two-cycles, we need to solve $(z^2+c)^2+c=z$. This is a degree 4 equation, but we already know two of the solutions—namely, the fixed points p_+ and p_- we already found. Dividing the polynomial $z^4+2cz^2-z+(c^2+c)$ by the polynomial z^2-z+c (the latter of which gives the equation for the fixed points), we get $z^2+z+(c+1)$. So the points on the two-cycles are the solutions to $z^2+z+(c+1)=0$. These are

$$q_\pm = \frac{-1\pm\sqrt{1-4(c+1)}}{2}.$$

To ensure that this is an *attractive* two-cycle, we need to require that

$$|Q_c'(q_-)|\cdot|Q_c'(q_+)|<1.$$

This becomes the inequality $|(-1+\sqrt{1-4(c+1)})(-1-\sqrt{1+4(c+1)})|<1$. Multiplying out, we find

$$|4(c+1)|<1$$

or

$$|c+1|<\frac{1}{4}.$$

The c-values satisfying this inequality form a disk with radius $\frac{1}{4}$, centered at -1.

3.3. **Attractive cycles of other periods.** It is harder to exactly describe all c values for which Q_c has an attractive 3-cycle, but we can find the centers for these regions without working too hard. We look for the c values which give an attractive 3-cycle *containing* 0. The orbit of 0 looks like

$$0\to c\to c^2+c\to(c^2+c)^2+c$$

and for this to be a 3-cycle we need

$$(c^2+c)^2+c=0.$$

Notice that this is a degree 4 equation, and so will have four complex solutions. Mathematica can find them numerically for us. It gives: $c=0$, $c=-1.75488$, $c=-0.1226\pm0.7449i$. The first solution, $c=0$, is the center for the period 1 part of the Mandelbrot set (the main cardioid). The remaining three c-values are the centers for the three period 3 decorations in the Mandelbrot set.

Exercise 3.4. Using a computer program that allows you to zoom in on pieces of the Mandelbrot set, locate the three period 3 decorations.

To find the centers of the period 4 decorations, we make 0 lie on a cycle of period 4. For this, c needs to satisfy the equation

$$((c^2+c)^2+c)^2+c=0.$$

This is a degree 8 equation, and so has eight complex solutions. One of them corresponds to the period 1 decoration (the main cardioid), and one of them corresponds to the period 2 decoration. This leaves six decorations which have period 4. Mathematica gives them as: $c=-1.3107$, $c=-1.9408$, $c=-0.1565\pm1.03225i$, $c=0.28227\pm0.53006i$.

Exercise 3.5. Repeat the previous exercise, but find the six decorations which have period 4.

For period 5 decorations we would solve a degree 16 equation. One solution corresponds to the period 1 decoration, leaving 15 decorations which have period 5.

For period 6 decorations we would solve a degree 32 equation. One solution corresponds to period 1, one solution corresponds to period 2, and three solutions correspond to period 3 (found above). This leaves 27 period 6 decorations.

You get the idea.

3.6. **Bulbs around the main cardioid.** Each bulb coming off the main cardioid can be labelled with a rational number $\frac{p}{q}$. The denominator is the *period* of the bulb, which can be identified in any of the following ways:

1. The period is the number of spokes, or 'antennas', near the top of the bulb (including the trunk which attaches to the main part of the bulb).
2. Looking in the Julia set for any point in the bulb, the period is the number of 'lobes' in a cluster.
3. Given any c in the bulb, and any point in the corresponding Julia set, it is the period of the cycle that the orbit converges to.

The numerator p can be identified in either of the following ways:

1. Look at the antennas near the top of the bulb. Starting from the main trunk and moving counterclockwise, count which antenna is the smallest. The number of this antenna is p (so if the second antenna is the smallest, $p = 2$).
2. Choose a c value in the bulb, compute the Julia set, and find the resulting attractive cycle. This cycle will jump from one lobe of the Julia set to another. Starting in one lobe, it will move counterclockwise—possibly skipping over other lobes in the process. The number p is the number of lobes you pass when going from one point of the cycle to the next.

We let $B_{p/q}$ denote the bulb corresponding to $\frac{p}{q}$ via the above rules. The bulbs on the main cardioid obey the following pattern:

Theorem 3.7. *Pick two bulbs $B_{p/q}$ and $B_{p'/q'}$. The largest bulb between them is $B_{(p+p')/(q+q')}$.*

The **Farey sum** (also called the **Farey child**) of $\frac{p}{q}$ and $\frac{p'}{q'}$ is defined to be

$$\frac{p}{q} \oplus \frac{p'}{q'} = \frac{p+p'}{q+q'}.$$

We can make sequences of rational numbers by starting with $\{\frac{0}{1}, \frac{1}{1}\}$ and constructing all their Farey progeny. After the first generation we get

$$\frac{0}{1}, \quad \frac{1}{2}, \quad \frac{1}{1},$$

then after the second generation we have

$$\frac{0}{1}, \quad \frac{1}{3}, \quad \frac{1}{2}, \quad \frac{2}{3}, \quad \frac{1}{1}.$$

Generation three gives

$$\frac{0}{1}, \quad \frac{1}{4}, \quad \frac{1}{3}, \quad \frac{2}{5}, \quad \frac{1}{2}, \quad \frac{3}{5}, \quad \frac{2}{3}, \quad \frac{3}{4}, \quad \frac{1}{1}.$$

And so on. These sequences are called **Farey sequences**. The above theorem says that if you start at the cusp of the main cardioid of the Mandelbrot set and move around counterclockwise, the patterns of the bulbs form a Farey sequence. The size of the bulbs is controlled by which generation they correspond to in the Farey sequences.

Exercise 3.8.

(a) Draw a picture of the main cardioid of the Mandelbrot set, and label all the period 2, 3, and 4 bulbs attached to it. Also label the bulb $B_{3/8}$.

(b) What equation would you need to solve in order to find all the centers of the period 8 decorations?

(c) How many period 8 decorations are there?

(d) Suppose I gave you a picture of the Mandelbrot set, together with three marked points lying in either the main cardioid or adjoining bulbs. I also give you three pictures of Julia sets. You should be able to match each Julia set to the point in the Mandelbrot set with which it corresponds.

(e) Explain the algorithm we use to compute the Mandelbrot set.

3.9. **Bifurcation points.** Each of the bulbs attached to the main cardioid is attached in exactly one point (if it doesn't look like this in a certain picture of the Mandelbrot set, it's because the picture was drawn using an algorithm with too few iterations). In the first part of the course we already studied the point where the period two bulb is attached—this is when $c = \frac{3}{4}$, and it is a period doubling bifurcation. The other points of attachment occur where an attractive fixed point becomes neutral and then splits into an attractive cycle. The point where $B_{1/3}$ is attached is a period *tripling* bifurcation, the point where $B_{2/5}$ is attached is a period *quintupling* bifurcation, etc. We have period scaling bifurcations of all orders.

Even without using computers, we can find out where these bifurcation points occur. We first recall that our fixed points are

$$p_{\pm} = \frac{1 \pm \sqrt{1-4c}}{2}.$$

We let $w = 1 - 4c$ and write w in polar form as $w = re^{i\theta}$. We then have

$$p_{\pm} = \frac{1 \pm \sqrt{r}e^{i\theta/2}}{2} = \frac{1 \pm \sqrt{r}\cos(\theta/2)}{2} \pm \frac{\sqrt{r}\sin(\theta/2)}{2} \cdot i.$$

Only one of these fixed points will be attractive. To understand which one, we need to be careful about the signs. So let's use the notation

$$p_u = \frac{1 + u\sqrt{r}e^{i\theta/2}}{2} = \frac{1 + u\sqrt{r}\cos(\theta/2)}{2} + u\frac{\sqrt{r}\sin(\theta/2)}{2} \cdot i$$

where u is either $+1$ or -1. We then have that

$$|Q_c'(p_u)|^2 = (1 + u\sqrt{r}\cos(\theta/2))^2 + (u\sqrt{r}\sin(\theta/2))^2 = 1 + 2u\sqrt{r}\cos(\theta/2) + r.$$

For an attractive fixed point we need $|Q_c'(p_u)| < 1$, and since r is positive this will only occur when $u\cos(\theta/2)$ is negative. So we have the following conclusion:

The attractive fixed point is p_u where u is $+1$ or -1 according to the rule that $u\cos(\theta/2)$ must be negative.

Now, there are two things we know about the c-value where the bifurcation point for $B_{p/q}$ occurs. These are:

(1) $r = 4\cos^2(\theta/2)$

(2) $Q_c'(p_u) = e^{2\pi(p/q)i}$.

The first comes from the fact that c must be on the boundary (the outer rim) of the main cardioid (recall from previous discussion that the equation for the main cardioid is $r < 4\cos^2(\theta/2)$). The second comes from discussion in class.

Plugging our equation for p_u into equation (2), we get

$$\cos(2\pi(p/q)) = 1 + u\sqrt{r}\cos(\theta/2) \qquad \text{and} \qquad \sin(2\pi(p/q)) = u\sqrt{r}\sin(\theta/2).$$

Equation (1) says that $\sqrt{r} = \pm 2\cos(\theta/2)$, but we can be more precise about the sign. Since $\sqrt{r}$ is necessarily positive, the sign must be $-u$ (since u is the sign which makes $u\cos(\theta/2)$ negative). So $\sqrt{r} = -2u\cos(\theta/2)$. Substituting this into the two equations above, we get

- $\cos(2\pi(p/q)) = 1 - 2\cos^2(\theta/2) = \cos(\theta)$
- $\sin(2\pi(p/q)) = -2\cos(\theta/2)\sin(\theta/2) = -\sin(\theta)$.

(we have used two trig identities: $\sin(2x) = 2\sin(x)\cos(x)$ and $\cos(2x) = 1 - 2\cos^2(x)$). These two equations completely determine θ.

If we assume for convenience that $\frac{p}{q} < \frac{1}{2}$, then the solutions to $\sin(\theta) = -\sin(2\pi(p/q))$ are $\pi(1 + 2\frac{p}{q})$ and $\pi(2 - 2\frac{p}{q})$. The only one of these which satisfies the equation $\cos(2\pi(p/q)) = cos(\theta)$ is $\pi(2 - 2\frac{p}{q})$. So $\theta = \pi(2 - 2\frac{p}{q})$.

What does this mean? It means that w lies on the ray of complex numbers whose argument is $2\pi - 2\pi(p/q)$. These are rays emanating from the cusp of the main cardioid. Translating to c, we find that c lies on the ray of complex numbers emanating from the cusp of the main cardioid which is an angle of $2\pi(p/q)$ radians measured *clockwise from the negative real axis*.

Summary: When $\frac{p}{q} \leq \frac{1}{2}$, the bulb $B_{p/q}$ is attached to the main cardioid at a bifucation point characterized as follows. If you draw the segment from the cusp of the main cardioid to this bifucation point, the angle from the segment to the negative real axis measures $2\pi(p/q)$ radians.