Math 457

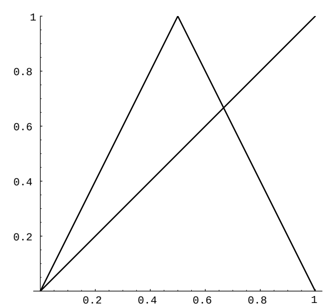
Homework Due Wednesday, April 20

1. Consider the doubling function $D\colon [0,1) \to [0,1)$. We know how to make lots of cycles for D—for instance, the point $a = \frac{1}{5} = [0.\overline{00011}]$ is part of a 5-cycle.
   (a) Give a rational number which is within a distance of 2^{-8} of a, and whose orbit eventually lands on the fixed point 0.
   (b) Give a rational number which is within a distance of 2^{-8} of a, and whose orbit eventually lands on $\frac{1}{3}$ (and therefore is eventually periodic, of period 2).
   (c) Give a rational number which is within a distance of 2^{-8} of a, and whose orbit eventually lands on a 3-cycle.
   (d) If I give you an integer $N \geq 1$ and any number $b \in [0,1)$, explain how you could find a point within a distance of 2^{-N} of a such that the orbit eventually lands on b.

   Remark: Notice the consequences of (d). Even though we completely understand what happens to the orbit of our seed value a, this doesn't mean we can predict what happens to orbits within some small distance of a. In fact, (d) shows that **anything** can happen to such orbits—no matter how small a distance we restric to. So this is an example of the 'chaos' phenomenon we talked about a couple weeks ago.

2. Consider the function $T\colon [0,1] \to [0,1]$ defined by $T(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2-2x & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$

   A graph is shown below:



   (a) Convert the binary number $[1.111\ldots]$ into base 10. The answer should be very simple.
   (b) If d is a binary digit (0 or 1), let $\tilde{d}$ denote the opposite: if $d = 0$ then $\tilde{d} = 1$, and if $d = 1$ then $\tilde{d} = 0$. Explain why, if we use binary expansions, T has the following description:

$$T(0.0a_1a_2a_3\ldots) = 0.a_1a_2a_3\ldots \qquad \text{and} \qquad T(0.1a_1a_2a_3\ldots) = 0.\tilde{a}_1\tilde{a}_2\tilde{a}_3\ldots.$$

   You might find (a) useful in your explanation.
   (c) Find the fixed points of T as well as all the 2-cycles and 3-cycles (it is okay to just give the binary expansions).
   (d) For each of the fixed points and cycles that you found, determine whether it's attractive or repulsive. (Hint: What do you know about the derivative of T?)
   (e) Show how to construct an n-cycle for T, given any integer $n \geq 2$ (note that I am not asking you to construct *all* n-cycles).

(f) Extra credit: Prove that every rational number is an eventually periodic point for T.

(g) If a is an irrational number, what will the orbit of a look like? Will it be eventually periodic? Will it converge to a cycle?

(h) Suppose you pick a random seed value a in the interval $[0, 1]$ and have a computer iterate T. If the computer only keeps track of a finite number of decimal places, what will happen to the orbit? How is this different from what *really* happens to the orbit? Explain.

3. Let $f(x) = x^2 + c$, and consider the orbit of 0. If $c = -1.2$, computer calculations shows that the orbit converges to the 2-cycle $\{0.170822, -1.17082\}$. If $c = -1.1$, the computer shows the orbit converging to the 2-cycle $\{0.091608, -1.09161\}$. Note that the numbers in each 2-cycle seem to add up to -1 (assuming the small discrepancy we're seeing is due to computer roundoff error). Explain this phenomenon using the theory from chapter 6.

4. Consider the family of functions $F_\lambda(x) = \lambda x(1 - x)$, where $\lambda > 0$ (called the 'logistic family'). It is a theorem that for each λ this function has at most one attracting cycle, and that the seed value $\frac{1}{2}$ will be attracted to it. Using everything you've learned so far and whatever technology you feel like, answer the following questions about the orbit of $\frac{1}{2}$. Note that we are only considering $\lambda > 0$.

   (a) Describe the orbit of $\frac{1}{2}$ under $F_{\frac{1}{3}}$.

   (b) Determine the smallest value of λ for which a bifurcation occurs. What type of bifurcation is it?

   (c) Determine where the next bifurcation occurs, and also identify its type.

   (d) Determine as best you can the smallest value of λ for which F_λ has an attracting 4-cycle.

   (e) Sketch the orbit diagram for F_λ in the range $0 < \lambda < 3.5$.

   (f) What is the smallest value of λ for which F_λ has an attracting 3-cycle? You might have to estimate this using a computer graph. Write down what equation you would solve in order to find λ exactly.

   [Note: On the exam you will have to answer questions like (a)-(c) without the use of a computer, and questions like (d)-(f) if I give you appropriate computer graphs. Although (d) can be solved exactly with just a little bit of work.]