Qualifying Examination

Theory of Probability

September 2004

NAME:__________________________

Instructions:

(1) This is a close book and close notes exam.

(2) This examination consists a total of six questions and comprises two printed pages.

(3) Answer all questions in 3 hours. Solve problems step by step and show all your work.
Throughout this paper, all random variables are real-valued unless otherwise specified.

1. (a) (9 points) Let $A_1, A_2, \ldots$ be a sequence of events on a probability space $(\Omega, \mathcal{F}, P)$.
   
   (i) Prove that if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A_n \text{ i.o.}) = 0$.

   (ii) Prove that if $\sum_{i=1}^{n} P(A_i) \to \infty$ and $\frac{\sum_{i=1}^{n} \sum_{j=1}^{n} P(A_i A_j)}{(\sum_{i=1}^{n} P(A_i))^2} \to 1$ as $n \to \infty$, then $P(A_n \text{ i.o.}) = 1$.

   (b) (5 points) Let $X_1, X_2, \ldots$ be independent distributed random variables with a common probability density function

   $$f(x) = \frac{c_0}{(1 + x^2) \ln(3 + x^2)} \text{ for } -\infty < x < \infty,$$

   where $c_0 > 0$ is a constant. Prove that

   $$\frac{1}{n} \sum_{i=1}^{n} X_i \to 0 \text{ in probability}.$$

2. (a) (10 points) Let $X_1, X_2, \ldots$ be independent and identically distributed random variables and let $S_n = \sum_{i=1}^{n} X_i$. Prove that

   $$\frac{S_n}{n} \to \mu \text{ a.s.}$$

   for some finite constant $\mu$ if and only if $E(|X_1|) < \infty$.

   (b) (8 points) Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with $E(X_1) = \mu$ and $\text{Var}(X_1) = \sigma^2 < \infty$. Put

   $$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i, \quad s_n = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2}, \quad T_n = n^{1/2} (\bar{X} - \mu)/s_n.$$

   Prove

   (i) $s_n/\sigma \to 1 \text{ a.s. as } n \to \infty$;

   (ii) $T_n \xrightarrow{d} N(0,1)$.

3. (a) (5 points) Assume that random vectors $(X, Y)$ and $(X, Z)$ have the same joint distribution. Prove that if $g : \mathbb{R}^1 \to \mathbb{R}^1$ is a measurable function with $E|g(Y)| < \infty$, then $E(g(Y) \mid X) = E(g(Z) \mid X)$.

   (b) (4 points) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with $E(|X_1|) < \infty$. Let $S_n = \sum_{i=1}^{n} X_i$. Compute $E(X_1 | S_n)$.

   (c) (4 points) Let $\{T_n, \mathcal{F}_n, n \geq 0\}$ be a submartingale and $N$ be a stopping time. Prove that $E(T_0) \leq E(T_{\min(n, N)}) \leq E(T_n)$ for $n \geq 1$. 

1
(d) (7 points) Let \( X_1, X_2, \ldots \) be independent random variables with \( P(X_i = 1) = p \) and \( P(X_i = -1) = 1 - p \), where \( 1/2 < p < 1 \). Let \( S_0 = 0 \) and \( S_n = \sum_{i=1}^{n} X_i \) for \( n = 1, 2, \ldots \), \( \mathcal{F}_n = \sigma(X_i \mid i \leq n) \) for \( \geq 1 \) and \( \mathcal{F} \) be the trivial \( \sigma \)-field.

(i) Let \( \phi(x) = \((1-p)/p)^x \). Prove that \( \{\phi(S_n), \mathcal{F}_n, n \geq 0\} \) is a martingale;

(ii) Let \( T_x = \inf\{n \geq 1 : S_n = x\} \). Prove that for all positive integer \( k \)

\[
P(T_{-k} < T_k) = \frac{1}{1 + \phi(-k)}.
\]

4. (14 points) Let \((X_n)_{n \geq 0}\) be a Markov chain on \(\{0, 1, 2\}\) with the transition matrix

\[
p = \begin{pmatrix}
0.5 & 0.25 & 0.25 \\
0.1 & 0.55 & 0.35 \\
0 & 0.5 & 0.5
\end{pmatrix}.
\]

(i) Which sets are closed? Which sets are irreducible? Which states are recurrent?

(ii) Find the stationary distribution \( \pi \) of \((X_n)_{n \geq 0}\);

(iii) Let \( T_x = \inf\{n \geq 1 : X_n = x\} \). Find \( E_x(T_x) \) for \( x = 0, 1, 2 \).

(iv) Find the initial distribution of \( X_0 \) so that \((X_n, n \geq 0)\) is a stationary Markov chain.

5. (14 points) Let \( \{B(t), t \geq 0\} \) be a Brownian motion and \( \mathcal{F}_t = \sigma(B(s), 0 \leq s \leq t) \).

(i) Let \( \tau \) be a finite stopping time with respect to \(\{\mathcal{F}_t, t \geq 0\}\) and let \( X(t) = B(t + \tau) - B(\tau) \).

Prove that \(\{X(t), t \geq 0\}\) is a Brownian motion and is independent of \( \tau \).

(ii) Prove that \(\{B(t), t \geq 0\}\) and \(\{B^2(t) - t, t \geq 0\}\) are martingales.

(iii) Let \( a > 0 \) and \( \tau = \inf\{t \geq 0 : B(t) \notin (-a, a)\} \). Prove that \( P(B(\tau) = a) = 1/2 \) and \( E(\tau) = a^2 \).

6. (18 points) Let \( X_1, X_2, \ldots \) be independent and identically distributed standard normal random variables and let \( S_n = \sum_{i=1}^{n} X_i \).

(i) Prove that for all \( x > 0 \)

\[
P\left(\max_{1 \leq i \leq n} S_i \geq x\sqrt{n}\right) \leq 2P(S_n \geq x\sqrt{n})
\]

and

\[
P\left(\max_{1 \leq i \leq n} |S_i| \geq x\sqrt{n}\right) \leq \frac{2}{x} e^{-x^2/2}.
\]

(ii) Prove that

\[
\lim_{n \to \infty} \sup_{n \geq 1} \frac{\max_{1 \leq i \leq n} |S_i|}{(2n \ln \ln n)^{1/2}} = 1 \text{ a.s.}
\]
Solutions

Qualifying Examination

Theory of Probability

September 2004
1. (a) (i) (4 points)

\[ P(A_{\infty}) = \lim_{n \to \infty} P(\bigcup_{m=n}^{\infty} A_m) \leq \lim_{n \to \infty} \sum_{m=n}^{\infty} P(A_m) = 0 \]

(ii) (5 points) Observe that

\[ P(\sum_{i=1}^{n} 1_{\{A_i\}} \leq (1/2) \sum_{i=1}^{n} P(A_i)) \]
\[ = P(\sum_{i=1}^{n} (1_{\{A_i\}} - P(A_i)) \leq -(1/2) \sum_{i=1}^{n} P(A_i)) \]
\[ \leq \frac{4}{(\sum_{i=1}^{n} P(A_i))^2} \text{Var}(\sum_{i=1}^{n} 1_{\{A_i\}}) \]
\[ = \frac{4}{(\sum_{i=1}^{n} P(A_i))^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \{P(A_iA_j) - P(A_i)P(A_j)\} \]
\[ \to 0 \]

as \( n \to \infty \) by the assumption. Hence

\[ P(\sum_{i=1}^{n} 1_{\{A_i\}} \geq (1/2) \sum_{i=1}^{n} P(A_i)) \to 1. \]

Since \( \sum_{i=1}^{n} P(A_i) \to \infty \), the above yields \( \sum_{i=1}^{n} 1_{\{A_i\}} \to \infty \) in probability and hence a.s. (because \( \sum_{i=1}^{n} 1_{\{A_i\}} \) is non-decreasing). Therefore, \( P(A_{\infty}) = 1 \).

1. (b) (5 points) Let \( \tilde{X}_i = X_i 1_{\{|X_i| \leq n\}} \) for \( i = 1, 2, \cdots, n \) and \( T_n = \sum_{i=1}^{n} \tilde{X}_i \). Then for \( \varepsilon > 0 \) with \( S_n = \sum_{i=1}^{n} X_i \)

\[ P(|S_n| \geq \varepsilon n) \leq P(\max_{1 \leq i \leq n} |X_i| > n) + P(|T_n| \geq \varepsilon n) \]
\[ \leq nP(|X_1| > n) + (\varepsilon n)^{-2} E(T_n^2) \]
\[ = 2n \int_{n}^{\infty} f(x)dx + 2(\varepsilon n)^{-2} \int_{0}^{n} x^2 f(x)dx \]
\[ \leq \frac{4c_0}{\ln(3 + n^2)} + 2c_0 (\varepsilon n)^{-2}n \]
\[ \to 0 \]

as \( n \to \infty \), as desired.
2. (a) **Sufficiency (7 points).** Assume that $E|X_i| < \infty$ and also assume $X_i \geq 0$. Let $\mu = EX_1$. Let $Y_i = X_11\{0 \leq X_i \leq i\}$ and $T_n = \sum_{i=1}^{n} Y_i$. Then $ET_n/n \to \mu$, $\sum_{i=1}^{\infty} P(X_i > i) \leq EX_1 < \infty$ and hence by the Borel-Cantelli lemma $P(X_n \neq Y_{n\text{i.o.}}) = 0$. It suffices to show that

$$\frac{T_n}{n} \to \mu \ a.s.$$ 

Observe that for $\varepsilon > 0$

$$P(|T_n - ET_n| \geq \varepsilon n) \leq (\varepsilon n)^{-2}\text{Var}(T_n) \leq \varepsilon^{-2} \frac{1}{n} EX_1^2 1_{\{X_1 \leq n\}}.$$

Let $n_k = [\theta^k]$, where $\theta > 1$. Then

$$\sum_{k=1}^{\infty} P(|T_{n_k} - ET_{n_k}| \geq \varepsilon n_k) \leq \varepsilon^{-2} \sum_{k=1}^{\infty} \frac{1}{n_k} EX_1^2 1_{\{X_1 \leq n_k\}}$$

$$\leq \varepsilon^{-2} \sum_{k=1}^{\infty} \frac{1}{n_k} \sum_{j=1}^{k} EX_1^2 1_{\{n_{j-1} < X_1 \leq n_j\}}$$

$$\leq c_1 \varepsilon^{-2} \sum_{j=1}^{\infty} \frac{1}{n_j} EX_1^2 1_{\{n_{j-1} < X_1 \leq n_j\}}$$

$$\leq c_2 \varepsilon^{-2} EX_1 < \infty$$

where $c_1, c_2$ are constants. Therefore by the Borel-Cantelli lemma

$$\frac{T_{n_k} - ET_{n_k}}{n_k} \to 0 \ a.s.$$ 

Thus

$$\frac{T_{n_k}}{n_k} \to \mu \ a.s.$$ 

For general $m$, let $n_{k-1} \leq m < n_k$. Then

$$\frac{T_{n_{k-1}}}{n_{k-1}} \leq \frac{T_m}{m} \leq \frac{T_{n_k}}{n_k}$$ 

and hence

$$\frac{\mu}{\theta} \leq \lim_{m \to \infty} \frac{T_m}{m} \leq \limsup_{m \to \infty} \frac{T_m}{m} \leq \theta \mu \ a.s.$$ 

This proves $T_n/n \to \mu$ a.s. by the arbitrariness of $\theta > 1$.

For the general case, write $X_i = X_i^+ - X_i^-$. Then, the result follows from the first case.

**Necessity. (3 points)** Assume that $S_n/n \to \mu$ a.s. Then

$$|X_n - \mu|/n \leq |(S_n - n\mu)|/n + |S_{n-1} - (n-1)\mu|/n \to 0 \ a.s.$$ 

By the Borel-Cantelli lemma, we have

$$\sum_{n=1}^{\infty} P(|X_n - \mu| \geq n) = \sum_{n=1}^{\infty} P(|X_1 - \mu| \geq n).$$

Hence $E|X_1 - \mu| < \infty$, which implies $E|X_1| < \infty$. 

2
2. (b) (i) (3 points) We have
\[ s_n^2 = \frac{1}{n-1} \left( \sum_{i=1}^{n} X_i^2 - \frac{n}{n-1} \bar{X}^2 \right) \rightarrow E X_1^2 - (E X_1)^2 = \sigma^2 \text{ a.s.} \]
by the law of large numbers.

(ii) (5 points) Write
\[ T_n = \frac{n^{1/2}(\bar{X} - \mu)}{\sigma} \frac{\sigma}{s_n} \]
By the central limit theorem
\[ \frac{n^{1/2}(\bar{X} - \mu)}{\sigma} \xrightarrow{d} N(0,1) \]
and by (i), \( \sigma/s_n \rightarrow 1 \) in probability, we have \( T_n \xrightarrow{d} N(0,1) \)

3. (a) (5 points)
Consider 4 different cases:

**Case 1.** \( g \) is an indicator function, i.e., \( g(x) = 1_{\{A\}}(x) \), where \( A \) is a Borel set. Then for any Borel set \( B \)
\[ E(g(Y)|\{X \in B\}) = E(1_{\{Y \in A\}}1_{\{X \in B\}}) = P(Y \in A, X \in B) \]
\[ = P(Z \in A, X \in B) = E(g(Z)1_{\{X \in B\}}) \]
Therefore \( E(g(Y)|X) = E(g(Z)|X) \).

**Case 2.** \( g \) is simple function, i.e., \( g = \sum_{i=1}^{n} c_i g_i \), where \( g_i \) are indicator functions. Then
\[ E(g(Y)|X) = \sum_{i=1}^{n} c_i E(g_i(Y)|X) = \sum_{i=1}^{n} c_i E(g_i(Z)|X) = E(g(Z)|X). \]

**Case 3.** \( g \) is non-negative. Then there exists non-decreasing simple functions \( g_n \) such that \( g_n \rightarrow g \). By the monotone convergence theorem,
\[ E(g(Y)|X) = \lim_{n \to \infty} E(g_n(Y)|X) = \lim_{n \to \infty} E(g_n(Z)|X) = E(g(Z)|X) \]

**Case 4.** general case. Write \( g = g^+ - g^- \). Then
\[ E(g(Y)|X) = E(g^+(Y)|X) - E(g^-(Y)|X) = E(g^+(Z)|X) - E(g^-(Z)|X) = E(g(Z)|X) \]

3. (b) (4 points) Since \( \{X_i, 1 \leq i \leq n\} \) are i.i.d., \( (X_1, S_n) \) and \( (X_i, S_n) \) have the same joint distribution for each \( i \). Hence \( E(X_1|S_n) = E(X_i|S_n) \) and
\[ S_n = E(S_n|S_n) = \sum_{i=1}^{n} E(X_i|S_n) = \sum_{i=1}^{n} E(X_1|S_n) = n E(X_1|S_n). \]
This shows that \( E(X_1|S_n) = S_n/n \).
3. (c) (4 points)

We have

\[ ET_{\min(n,N)} = ET_N 1\{N < n\} + ET_n 1\{N \geq n\} = \sum_{j=1}^{n-1} ET_j 1\{N = j\} + E(T_n 1\{N \geq n\}) \]

Since \(\{T_n, \mathcal{F}_n, n \geq 0\}\) is a submartingale, \(E(T_j 1\{N = j\}) \leq E(T_n 1\{N = j\})\) for \(j \leq n - 1\). Hence

\[ \sum_{j=1}^{n-1} ET_j 1\{N = j\} + E(T_n 1\{N \geq n\}) \leq \sum_{j=1}^{n-1} E(T_n 1\{N = j\}) + E(T_n 1\{N \geq n\}) = ET_n \]

which proves that \(ET_{\min(n,N)} \leq ET_n\). Noting that \(\{N \geq n\} \in \mathcal{F}_{n-1}\), we have

\[ E(T_n 1\{N \geq n\}) \geq E(T_{n-1} 1\{N \geq n\}) \]

and hence

\[ ET_{\min(n,N)} \geq ET_N 1\{N < n\} + E(T_{n-1} 1\{N \geq n\}) = ET_{\min(n-1,N)} \geq \cdots \geq ET_{\min(0,N)} = ET_0. \]

3. (d) (i) (3 points) We have

\[ E(\phi(S_n)|\mathcal{F}_{n-1}) = \phi(S_{n-1})E(\phi(X_n)) = \phi(S_{n-1}) \]

(ii) (4 points). Let \(N = T_{-k} \wedge T_k\). Then \(N\) is a stopping time and

\[
P(N > 2mk) = P(|S_i| < k, i = 1, \ldots, 2mk) \\
\leq P(|S_{2ik} - S_{2(i-1)k}| < 2k, i = 1, \cdots m) \\
= P(|S_{2k}| < 2k)^m = \left(1 - (p^{2k} + (1-p)^{2k})\right)^m
\]

Hence \(E(N) < \infty\). Note that \(\phi(S_{n\wedge N})\) is bounded, we have

\[ \phi(0) = E\phi(S_N) = P(T_{-k} < T_k)\phi(-k) + P(T_k < T_{-k})\phi(k) \]

Using \(P(T_{-k} < T_k) + P(T_k < T_{-k}) = 1\) and solving the above equation gives the desired result.
4. (i) (2 points)
\{0, 1, 2\} is closed, irreducible, and all states are recurrent.

(ii) (6 points)
Let \( \pi \) be the stationary distribution. Then \( \sum_{i=0}^{2} \pi(i)p(i, j) = \pi(j) \) for \( j = 0, 1, 2 \). Solving

\[
\begin{align*}
0.5\pi(0) + 0.1\pi(1) &= \pi(0), \\
0.25\pi(0) + 0.55\pi(1) + 0.5\pi(2) &= \pi(1), \\
0.25\pi(0) + 0.35\pi(1) + 0.5\pi(2) &= \pi(2), \\
\pi(0) + \pi(1) + \pi(2) &= 1
\end{align*}
\]

gives \( \pi(0) = 0.1, \pi(1) = 0.5, \pi(2) = 0.4 \).

(iii) (3 points) \( E_x(T_x) = \frac{1}{\pi(x)} \) = 10 for \( x = 0 \), = 2 for \( x = 1 \) and = 2.5 for \( x = 2 \).

(iv) (3 points) \( P(X_0 = i) = \pi(i) \) for \( i = 0, 1, 2 \). That is, \( P(X_0 = 0) = 0.1, P(X_0 = 1) = 0.5, P(X_0 = 2) = 0.4 \).

5. (i) (6 points)
Let \( n \geq 4 \) and define

\[ \tau_n = \frac{k}{n} \text{ if } \frac{k-1}{n} \leq \tau < \frac{k}{n}, \quad k = 1, 2, \ldots \]

Then \( \tau_n \) are stopping times. For \( t_i > 0 \) and Borel sets \( A_i \) and \( B \)

\[
P(B(t_i + \tau_n) - B(\tau_n) \in A_i, 1 \leq i \leq m, \tau \in B) \\
= \sum_{k=1}^{\infty} P(B(t_i + k/n) - B(k/n) \in A_i, 1 \leq i \leq m, (k-1)/n \leq \tau < k/n, \tau \in B) \\
= \sum_{k=1}^{\infty} P(B(t_i + k/n) - B(k/n) \in A_i, 1 \leq i \leq m, (k-1)/n \leq \tau < k/n, \tau \in B) \\
= \sum_{k=1}^{\infty} P(B(t_i) \in A_i, 1 \leq i \leq m)P((k-1)/n \leq \tau < k/n, \tau \in B) \\
= P(B(t_i) \in A_i, 1 \leq i \leq m)P(\tau \in B).
\]

Letting \( n \to \infty \) yields

\[
P(B(t_i + \tau) - B(\tau) \in A_i, 1 \leq i \leq m, \tau \in B) = P(B(t_i) \in A_i, 1 \leq i \leq m)P(\tau \in B).
\]

This proves that \( \{B(t + \tau) - B(\tau), t \geq 0\} \) is a Brownian motion and is independent of \( \tau \).

(ii) (4 points) For \( 0 < s < t \)

\[
E(B(t)|F_s) = B(s) + E(B(t) - B(s)|F_s) = B(s) + E(B(t) - B(s)) = B(s)
\]

5
Hence \( \{B(t), t \geq 0\} \) is a martingale.

Similarly, for \( 0 < s < t \)

\[
E(B(t) - t | F_s) = E((B(s) + B(t) - B(s))^2 | F_s) - t \\
= B^2(s) + 2B(s)E(B(t) - B(s)) + E((B(t) - B(s))^2 - t = B^2(s) - s.
\]

This proves that \( \{B^2(t), t \geq 0\} \) is also a martingale

(iii) (4 points)

Since \( \lim_{t \to \infty} B(t)/\sqrt{t} = \infty \) and \( \lim \inf_{t \to \infty} B(t)/\sqrt{t} = -\infty, \tau < \infty \) a.s., we have \( EB(\tau \wedge t) = 0 \) for any \( t > 0 \). Letting \( t \to \infty \) and using the bounded convergence theorem gives \( EB(\tau) = 0 \), which combines \( P(B(\tau) = a) + P(B(\tau) = -a) = 1 \) yields \( P(B(\tau) = a) = 1/2 \). Since \( B^2(t) - t \) is a martingale, we have

\[
E(\tau) = EB^2(\tau) = a^2(1/2) + (-a)^2(1/2) = a^2.
\]

6. (i) (9 points) Let \( A_k = \{S_j < x\sqrt{n}, 1 \leq j < k, S_k \geq x\sqrt{n}\} \). Then \( A_k \) are disjoint and \( \{\max_{1 \leq k \leq n} S_k \geq x\sqrt{n}\} = \bigcup_{1 \leq k \leq n} A_k \). We have

\[
P(\max_{1 \leq k \leq n} S_k \geq x\sqrt{n}) \leq P(S_n \geq x\sqrt{n}) + \sum_{1 \leq k \leq n} P(A_k, S_n < x\sqrt{n}) \\
\leq P(S_n \geq x\sqrt{n}) + \sum_{1 \leq k \leq n} P(A_k, S_n - S_k < 0) \\
= P(S_n \geq x\sqrt{n}) + \sum_{1 \leq k \leq n} (1/2)P(A_k) \\
= P(S_n \geq x\sqrt{n}) + (1/2)P(\bigcup_{1 \leq k \leq n} A_k) \\
= P(S_n \geq x\sqrt{n}) + (1/2)P(\max_{1 \leq k \leq n} S_k \geq x\sqrt{n})
\]

Hence

\[
P(\max_{1 \leq k \leq n} S_k \geq x\sqrt{n}) \leq 2P(S_n \geq x\sqrt{n})
\]

and

\[
P(\max_{1 \leq k \leq n} |S_k| \geq x\sqrt{n}) \leq P(\max_{1 \leq k \leq n} S_k \geq x\sqrt{n}) + P(\max_{1 \leq k \leq n} (-S_k) \geq x\sqrt{n}) \\
\leq 4P(S_n \geq x\sqrt{n}) \\
= \frac{4}{\sqrt{2\pi n}} \int_x^\infty e^{-t^2/2} dt \\
\leq 2 \int_x^\infty (t/x)e^{-t^2/2} dt = \frac{2}{x}e^{-x^2/2}
\]

(ii) (9 points)

Upper bound (5 points)
Let

\[ T_n = \max_{1 \leq i \leq n} \left| S_i \right| \left( 2n \ln \ln n \right)^{1/2} \]

For fixed \( \theta > 1 \), let \( n_k = \lfloor \theta^k \rfloor \). For \( \varepsilon > 0 \), by (i)

\[ P(T_{n_k} \geq 1 + \varepsilon) \leq C \exp(- (1 + \varepsilon)^2 \ln n_k) \leq C (\ln n_k)^{- (1 + \varepsilon)^2}, \]

which is summable. Hence by the Borel-Cantelli lemma

\[ \lim_{k \to \infty} \sup_{n_k} T_{n_k} \leq 1 \text{ a.s.} \]

Now \( m \) large, choose \( k \) so that \( n_{k-1} < m \leq n_k \). Then

\[ \lim_{m \to \infty} \sup_{m} T_{m} \leq \lim_{k \to \infty} \sup_{n_k} \theta^{1/2} T_{n_k} \leq \theta^{1/2} \text{ a.s.} \]

This proves that

\[ \lim_{m \to \infty} \sup_{m} T_{m} \leq 1 \text{ a.s.} \]

by the arbitrariness of \( \theta > 1 \).

**Lower bound** (4 points)

To prove the lower bound of \( \limsup \), let \( n_k = e^k \ln k \). Then

\[
\begin{align*}
\limsup_{k \to \infty} \frac{S_{n_k}}{(2n_k \ln \ln n_k)^{1/2}} &\geq \limsup_{k \to \infty} \frac{S_{n_k} - S_{n_{k-1}}}{(2n_k \ln \ln n_k)^{1/2}} - \limsup_{k \to \infty} \frac{|S_{n_{k-1}}|}{(2n_k \ln \ln n_k)^{1/2}} \\
&= \limsup_{k \to \infty} \frac{S_{n_k} - S_{n_{k-1}}}{(2n_k \ln \ln n_k)^{1/2}} - \limsup_{k \to \infty} \frac{\sqrt{n_{k-1}}}{\sqrt{n_k}} \frac{|S_{n_{k-1}}|}{(2n_{k-1} \ln \ln n_{k-1})^{1/2}} \\
&= \limsup_{k \to \infty} \frac{S_{n_k} - S_{n_{k-1}}}{(2(n_k - n_{k-1}) \ln \ln n_k)^{1/2}}.
\end{align*}
\]

It is known that for a standard normal random variable \( Z \)

\[ P(Z > x) \approx \frac{1}{x} e^{-x^2/2} \text{ as } x \to \infty \]

Hence for any \( 0 < \varepsilon < 1/2 \) and \( k \) large enough

\[
P\left(\frac{S_{n_k} - S_{n_{k-1}}}{(2(n_k - n_{k-1}) \ln \ln n_k)^{1/2}} > (1 - \varepsilon)\right) \approx \frac{1}{\ln n_k} \exp(- (1 - \varepsilon)^2 \ln n_k)
\]

\[ \approx \frac{1}{(\ln k)(\ln k)(1-\varepsilon)^2} \]

which is not summable. Note that \( \{S_{n_k} - S_{n_{k-1}}, k \geq 1\} \) are independent. By the Borel-Cantelli lemma,

\[ \limsup_{k \to \infty} \frac{S_{n_k} - S_{n_{k-1}}}{(2(n_k - n_{k-1}) \ln \ln n_k)^{1/2}} \geq 1 \text{ a.s.} \]

This proves the lower bound by the inequalities above.