DIMENSION OF CERTAIN CLEFT BINOMIAL RINGS

by

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Let R be an artinian ring with identity. Denote by $J = J(R)$ the Jacobson radical of R. The ring R is *cleft* if there is a subring $S \subseteq R$ such that $R = S \bigoplus J$ as abelian groups and $S \cong R/J$ as rings. Among the examples of cleft rings are finite dimensional algebras and all square-free rings.

The results of Anick and Green allow for the construction of projective resolutions of simple modules over finite dimensional split algebras of the form $K\Gamma/I$. The resulting bound on the projective dimension of simples provides a bound on the global dimension of the algebra. In this paper, we develop the necessary ideas to apply this method to certain members of the class of *binomial rings*. This class, defined recently by Sklar, includes the classes of monomial rings, square-free rings, and binomial algebras. Applied to arbitrary square-free rings, our technique gives a bound on global dimension by constructing a related monomial ring.

Lastly, we explore the above technique applied to rings that do not necessarily have identities. In their place, we consider *rings with local units*, that is rings in which every pair of elements have a common left and right identity. By extending the definition of binomial rings to certain rings with local units, we show our technique of constructing projective resolutions for simple modules is still valid. In this setting, the global dimension need not be determined by the projective dimension of the simple modules. However, we give conditions under which it is possible to find bounds for global dimension.

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CHAPTER I

INTRODUCTION

Let R be an associative ring with identity element 1_R and let $J = J(R)$ denote the *Jacobson radical* of R, that is, the intersection of all maximal left or right ideals of R. The ring R is said to be *left (right) artinian* if it satisfies the descending chain condition on left (right) ideals. We say R is *artinian* if it is both left and right artinian. For R artinian, J is the unique largest nilpotent ideal of R and we call the least nonnegative integer ℓ such that $J^\ell = 0$ the *Loewy length* of J. As a consequence of the artininian condition, every quotient

$$J^i/J^{i+1}$$

is a semisimple left (right) R-module. Further, R contains a *complete set of idempotents*, that is a finite set $\{e_1, e_2, \ldots, e_n\}$ of primitive, pairwise orthogonal idempotents in R such that

$$e_1 + e_2 + \ldots + e_n = 1_R.$$

Given such a complete set

$$Re_1,\ Re_2, \ldots, Re_n \quad \text{and} \quad e_1R,\ e_2R, \ldots, e_nR$$

represent all isomorphism classes of indecomposable, projective left and right R-modules (possibly with redundancies). It follows similarly,

$$Re_1/Je_1,\ Re_2/Je_2, \ldots, Re_n/Je_n \quad \text{and} \quad e_1R/e_1J,\ e_2R/e_2J, \ldots, e_nR/e_nJ$$

represent all isomorphism classes of simple left and right R-modules. We shall write the simple left and right modules more succinctly by $\overline{Re_i}$ and $\overline{e_iR}$. The ring R is basic if $Re_i \cong Re_j$ implies $i = j$ and we note that since every artinian ring is Morita equivalent to a basic ring, we often restrict our attention to basic rings. A subset $\{e_1, e_2, \ldots, e_k\}$ of a complete set such that $Re_1, \ldots, Re_k$ is an irredundant set of representatives of all indecomposable projective left R-modules is called a *basic set.*

As is customary, we investigate an artinian ring R by understanding the left and right R-modules. We denote by R-Mod (Mod-R) the category of all unital left (right) R-modules. Similarly, we write R-mod (mod-R) for the category of all finitely generated unital left (right) R-modules. We focus primarily on mod-R and note that symmetric statements can be made for R-mod. For a right module M_R, the *projective dimension* of M is the smallest integer k for which M_R has a projective resolution of length k. When such an integer exists, we write pdim $(M_R) = k$, otherwise we say pdim $(M_R) = \infty$. The global dimension of R, denoted gldim (R), is defined

$$\text{gldim } R = \max\{\text{pdim}(M) : M \in \text{mod-}R\}.$$

For R artinian, all finitely generated modules have composition series and this yields

$$\text{gldim } R = \max\{\text{pdim}(\overline{e_iR}) : i = 1, 2, \ldots n\}.$$

Given a module M, there are many straightforward methods for constructing a projective resolution for M, but these resolutions are usually not minimal with respect to length. Unfortunately, minimal resolutions are often difficult to construct. We therefore seek a method that balances ease of construction while producing a resolution that is close to minimal.

Some particularly useful tools present with a basic artinian ring R are two associated directed graphs called the *left and right quivers of* R. Let $\{e_1, e_2, \ldots, e_n\}$ be a basic set for R and let h_{ij} be the number of factors of e_jJ/e_jJ^2 that are isomorphic to $\overline{e_iR}$. The right quiver of R, denoted by Γ, is a directed graph with vertex set $\{e_1, e_2, \ldots, e_n\}$ and h_{ij} arrows starting at e_i and ending at e_j. The left quiver of R is defined analgously. Associated to every quiver is a semigroup structure on the set $P = P^* \cup \{\theta\}$, where P^* denotes all paths in Γ and θ indicates an element of the set that acts as zero in the semigroup. Such a P is an example of a *path semigroup.* Since the quivers of R are determined by the composition factors of J/J^2, much information about R can be understood using Γ and P.

Recall that given any field K and any quiver Γ with associated path semigroup P, we may construct $K\Gamma$, the *K-path algebra of* Γ. We denote by Rad $(K\Gamma)$ the Jacobson radical of $K\Gamma$. A finite-dimensional K-algebra is said to *split* if the endomorphism ring of every simple A-module is isomorphic to K. Gabriel's Theorem states given a finite-dimensional, split, basic algebra A with right quiver Γ, we have

$$A \cong K\Gamma/I$$

where I is an ideal of $K\Gamma$ and $(\text{Rad } K\Gamma)^\ell \subseteq I \subseteq (\text{Rad } K\Gamma)^2$ for some integer $\ell \geq 2$. Such an ideal I is said to be *adequate*. Much study of finite-dimensional algebras has been focused on adequate ideals. Adequate ideals generated by paths give rise to *monomial algebras* [8], while those generated by elements of the form $p - \lambda q$, $p, q \in P^*$, $\lambda \in K$ lead to *binomial algebras* [10]. Because of the intimate connection with the right quiver, one would hope that the global dimension of $A \cong K\Gamma/I$ would be determined entirely by Γ. It is known that this not the case [9]; there are examples of finite-dimensional algebras with the same quiver whose global dimension depends upon the ground field.

Despite the fact that even in well-understood examples, global dimension is not determined by Γ, the quiver can be used to study homological dimension of finite dimensional algebras. Using quivers, Anick and Green [4] gave a method for constructing projective resolutions of simples over $A \cong K\Gamma/I$. Although this method does not, in general, produce minimal resolutions, it is useful in finding upper bounds on projective dimension of simple modules. This leads to reasonable bounds on global dimension for some examples of finite-dimensional, split, basic algebras. The approach taken by Anick and Green relies heavily on properties of the associated path semigroup P and the connection with $A \cong K\Gamma/I$.

In this disseration, we extend the method of Anick and Green to certain members of the class of *cleft binomial rings*. All binomial algebras are binomial rings and in general, binomial rings have properties related to those of binomial algebras. In particular, a binomial ring R has a right quiver Γ and associated path semigroup P for which there is a very strong connection between P and R. A ring R is *cleft* if there is a semisimple subring $S \subseteq R$ such that $S \cong R/J$ as rings and

$$R = S \bigoplus J$$

as abelian groups. All finite dimensional algebras are cleft, but a cleft ring need not have an algebra structure (see [6]). Applying the method of Anick and Green to certain cleft binomial algebras, we are able to construct projective resolutions for simple modules. For many examples, this allows us to obtain bounds on the projective dimension of simple modules, which in turn, leads to upper bounds on global dimension.

We focus our attention on monomial algebras and their connection with certain cleft binomial rings. It has been shown (in [5]) that the global dimension of a monomial algebra does not depend upon the characteristic of the ground field, and there are a number of algorithms that may be used to compute this dimension. Therefore, if we can relate the dimension of a ring to that of a monomial algebra, we may be able to calculate the global dimension of the ring. For certain cleft binomial rings there are associated monomial algebras where the global dimension of the algebra gives a reasonable bound for the global dimension of the ring. We then compute an upper bound on global dimension for several examples of cleft binomial rings.

Finally, we conclude by considering a class of rings that need not have an identity.

A set $\mathcal{U} \subset R$ of nonzero idempotents is a *set of local units* for R if for each pair $x, y \in R$ there exists $u \in \mathcal{U}$ such that $x, y \in uRu$. Although R need not be unital, if $\mathcal{U}$ is a set of local units for R, then each uRu with $u \in \mathcal{U}$ is a unital ring. If R has a set of local units $\mathcal{U}$ then we call R a *ring with local units.* We impose certain conditions on R and $\mathcal{U}$ so that each uRu is a unital artinian ring. Similar to the unital case, we define a *cleft locally binomial ring* to be a type of ring with local units with properties related to cleft binomial rings. It is unknown if a cleft locally binomial ring R has a defined quiver, but despite this we may still find an associated path semigroup P. We then demonstrate that certain cleft locally binomial rings retain enough properties of cleft binomial rings so that the method of constructing projective resolutions of simple modules extends over certain cleft locally artinian rings. Although for a general cleft locally binomial ring R, global dimension need not be determined by the projective dimension of simples, we give conditions when this is the case.

CHAPTER II

PRELIMINARIES

II.1. Semigroups

We begin by exploring various properties of certain types of semigroups.

Given any finite directed graph Γ, there is a natural semigroup associated to Γ, called the *path semigroup of* Γ. When a directed graph Γ is related to a ring R (such as through the notion of a quiver), there are many interesting connections between the corresponding path semigroup P and R. We begin by generalizing the notion of a path semigroup.

Let P be a semigroup with operation $\cdot$ and zero element θ. We say P has an *identity set* if there is a subset $P_0 \subseteq P$ of orthogonal idempotents such that

$$P = \bigcup_{v_i, v_j \in P_0} v_i P v_j.$$

For such semigroups, a subset $X \subset P$ disjoint from $P_0 \cup \{\theta\}$ is said to be *prime* if for all $x \in X$, $x = a \cdot b$ for $a, b \in P$ implies $a \in P_0$ or $b \in P_0$. Notice, if X is a prime subset of P, and $x = a \cdot b \in X$ then we have $a \in P_0$ or $b \in P_0$, but not both. Henceforth, we will denote prime subsets of a semigroup P by P_1.

Definition II.1. Let P be a semigroup with an identity set P_0. We say P is a *path semigroup* if there is a prime subset $P_1 \subset P$ such that P is generated by $P_0 \cup P_1$ as a semigroup.

This definition of path semigroup is consistent with the previous notion. Given a directed graph Γ, the associated path semigroup has a set of orthogonal idempotents P_0 corresponding to the vertices of Γ. Indeed, since every path has an initial and terminal vertex we have

$$P = \bigcup_{v_i, v_j \in P_0} v_i P v_j.$$

Furthermore, the collection of arrows of Γ forms a prime subset P_1 of P. Clearly, P is generated as a semigroup by $P_0 \cup P_1$.

Adopting the previous established terminology, we call the subsets P_0 and P_1 the *vertices* and *arrows* of a path semigroup P. Denote by P^* the set $P \setminus \{\theta\}$. General elements of P^* will be called *paths.* Since P_0 is an identity set, we often refer to the vertices as *trivial paths.* By definition, a nontrivial path $p \in P^*$ is a sequence of arrows $p = (a_1 \ , a_2, \ldots, a_k)$ such that

$$\tau_P(a_i) = \sigma_P(a_{i+1}), \ i = 1, \ldots, k-1.$$

Adopting conventional notation, we will write such paths as $p = a_1 \cdot a_2 \cdots a_k$. Since

$$P_1 = \bigcup_{v_i, v_j \in P_0} v_i P_1 v_j,$$

we have set functions $\sigma_P, \tau_P : P_1 \longrightarrow P_0$ where, for $a \in P_1$, $\sigma(a), \tau(a) \in P_0$ are such so that $a = \sigma(a) \cdot a \cdot \tau(a)$. Since P_1 generates P, the set functions σ_P and τ_P extend in the natural way. Often of particular interest are paths with a certain initial (or left) vertex. We write ${}_iP$ to be the set of all paths p in P^* such that $p = v_i \cdot p$. For any subset $X \subseteq P^*$, write ${}_iX$ for $X \cap {}_iP$.

A *free path semigroup* is a path semigroup P with operation $\cdot$ and zero element θ such that:

(a) Each path $p \in P^* \setminus P_0$ can be written uniquely as a product

$$p = a_1 \cdot a_2 \cdots a_\ell$$

with $a_i \in P_1$ for all i.

(b) For all $p_1, p_2 \in P^*$ with $\tau(p_1) = \sigma(p_2)$ we have $p_1 \cdot p_2 \in P^*$.

We now explore how to create certain path semigroups starting with sets. Let X_0 and X_1 be disjoint sets. We say P is a *path semigroup generated by* $(X_0,\ X_1)$ if P is a path semigroup and there is a set bijection $\iota_P : X_0 \cup X_1 \longrightarrow P_0 \cup P_1$ such that $\iota_P(X_0) = P_0$ and $\iota_P(X_1) = P_1$. Since the semigroup structure of P depends upon σ_P and τ_P, two semigroups generated by $(X_0,\ X_1)$ need not be isomorphic as semigroups. When no confusion can arise, we often suppress the bijection ι_P and say P is generated by $(P_0,\ P_1)$ or simply say $P_0 \cup P_1$ generates P.

A *free path semigroup generated by* $(X_0,\ X_1)$ is a path semigroup P generated by $(X_0,\ X_1)$ that is also a free path semigroup. Again, when σ_P and τ_P are well understood, we often suppress the implied set bijections and say a free path semigroup is generated by $(P_0,\ P_1)$.

Definition II.2. Let P and Q be two path semigroups generated by $(X_0,\ X_1)$. A *path semigroup morhpism* $\phi : P \longrightarrow Q$ is a semigroup morphism that sends θ_P to θ_Q and such that for all $x \in X_0 \cup X_1$

$$\phi \circ \iota_P(x) = \iota_Q(x).$$

It follows path semigroup morphisms exist between path semigroups P and Q only when $\sigma_P \circ \iota_P = \sigma_Q \circ \iota_Q$ and $\tau_P \circ \iota_P = \tau_Q \circ \iota_Q$.

This leads to the following lemma:

Lemma II.3. *Given any path semigroup Q generated by $(X_0,\ X_1)$, there exists a free path semigroup P generated by $(X_0,\ X_1)$ and a unique path semigroup epimorphism $\phi : P \longrightarrow Q$.*

Proof. The proof of this result is essentially obvious.

Set $\sigma_P = \sigma_Q$, $\tau_P = \tau_Q$ and define the set P^* to be all words on $X_0 \cup X_1$. Let $P = P^* \cup \theta_P$. If the set P is endowed with an operation $\cdot$ given by concatenation of words subject to σ_P and τ_P, it follows easily that P is a free path semigroup generated by $(X_0,\ X_1)$. For $x \in X_0 \cup X_1$, define a morphism ϕ by

$$\phi(\iota_P(x)) = \iota_Q(x).$$

This map extends to all of P^* using the fact that all nonzero paths in P may be written as products of $\iota_P(x)$, with $x \in P_1$. Define $\phi(\theta_P) = \theta_Q$. Now ϕ is defined on all of P and an easy check shows ϕ is a well-defined path semigroup morphism.

If $q = \iota_Q(x_1) \cdots \iota_Q(x_k)$ then clearly $q = \phi(\iota_P(x_1) \cdots \iota_P(x_k))$ and ϕ is surjective. Let ψ be some other morphism. Then $\psi \circ \iota_P(x) = \iota_Q(x) = \phi \circ \iota_P(x)$ for all $x \in X_0 \cup X_1$. Then easily

$$\begin{aligned}\psi(\iota_P(x_1) \cdot \iota_P(x_2) \cdots \iota_P(x_k)) &= \psi(\iota_P(x_1)) \cdot \psi(\iota_P(x_2)) \cdots \psi(\iota_P(x_k)) \\ &= \phi(\iota_P(x_1)) \cdot \phi(\iota_P(a_2)) \cdots \phi(\iota_P(x_k)) \\ &= \phi(\iota_P(x_1) \cdot \iota_P(x_2) \cdots \iota_P(x_k))\end{aligned}$$

Hence, $\psi = \phi$. □

Corollary II.4. *If P and $\hat{P}$ are free path semigroups generated by $(X_0,\ X_1)$ with $\sigma_P = \sigma_{\hat{P}}$ and $\tau_P = \tau_{\hat{P}}$, then $P \cong \hat{P}$.* □

In light of the above lemmas, we will focus much of our attention on free path semigroups on sets.

Let P be a path semigroup on $(X_0,\ X_1)$. Let $\mathfrak{J} = P \setminus P_0$. The subset $\mathfrak{J}^n$ is defined as the n-fold product of elements from $\mathfrak{J}$ and is an ideal of P. Clearly, $\mathfrak{J}^n$ is generated by $(P_1)^n$. We say P is an *algebra semigroup* if

$$\bigcap_{i=1}^{\infty} \mathfrak{J}^i = \theta.$$

Properties of finite algebra semigroups were explored by Fuller in [7] and by Sklar in [10].

For the remainder of this thesis, a path semigroup P is always a path semigroup generated by a certain $(X_0,\ X_1)$. Unless specifically stated otherwise, we shall also take P to mean a free path semigroup.

For any free path semigroup we may define a function $v \longrightarrow 0$ for all $v \in P_0$ and $a \longrightarrow \ell(a) = 1$ for all $a \in P_1$. Since P is generated by P_1, this map extends to all of P in the following way. For each $p \in P^*$ where $p = a_1 \cdot a_2 \cdot \cdots \cdot a_k$ with $a_i \in P_1$, define

$$\ell(p) = \sum_{i=1}^{k} \ell(a_i) = k.$$

This makes P a *semigroup with length*, and this length function agrees with the standard length function used with semigroups associated to directed graphs.

II.2. Orderings

Throughout this section we will assume P is a free path semigroup generated by $(P_0,\ P_1)$. Our goal is to extend some of the key facts about certain partially ordered semigroups used in [4] to free path semigroups.

Definition II.5. A free semigroup P is *partially ordered* by $\leq$ if $\leq$ is a partial order on P^* and for $p_1, p_2, q_1, q_2 \in P^*$ with $p_1 \geq p_2$ we have

$$q_1 \cdot p_1 \geq q_1 \cdot p_2 \qquad \text{whenever} \quad q_1 \cdot p_1,\ q_1 \cdot p_2 \in P^*,$$

$$p_1 \cdot q_2 \geq p_2 \cdot q_2 \qquad \text{whenever} \quad p_1 \cdot q_2,\ p_2 \cdot q_2 \in P^*.$$

Definition II.6. An ordering $<$ on a partially ordered semigroup P is *suitable* if and only if each ${}_iP$ is well-ordered, with minimal element v_i.

Each free path semigroup P admits at least one, and in general many, suitable orderings. Indeed, for each $v_i \in P_0$, let $\leq_i^*$ be a well-ordering of ${}_iP \cup \{v_i\}$ with v_i

least. Let $\leq^*$ be the disjoint union of the $\leq_i^*$ with $v_i \in P_0$. Then $\leq^*$ is a partial order on the set $P_0 \cup P_1$. For each $v_i \in P_0$, define the *lexicographic order* $\leq_i$ on ${}_iP^*$ as follows:

> If $p = v_i \cdot a_1 \cdots a_h,\ q = v_i \cdot b_1 \cdots b_k \in P^*$ with all $a_i,\ b_j \in P_1$, then $p <_i q$ if $h < k$ or if $h = k$, and there is some $n \leq h$ with $a_i = b_i$ for all $1 \leq i \leq n-1$ and $a_n <^* b_n$.

Clearly, each $\leq_i$ is a well-ordering of ${}_iP^*$ for each $v_i \in P_0$. Finally, let $\leq$ denote the disjoint union of the orders $\leq_i$, so $\leq$ is a partial order on the set P^*. For this partial order we have the following:

Lemma II.7. *The partial order $\leq$ on P^* is a suitable order on the free path semigroup P.*

Proof. It suffices to show that P is partially ordered as a free path semigroup by $\leq$. But clearly, whenever $p_1 \leq p_2$ and $p_1 \cdot q,\ p_2 \cdot q \in P^*$ we must have $p_1 \cdot q \leq p_2 \cdot q$. Similarly, when $p_1 \leq p_2$ and $q \cdot p_1,\ q \cdot p_2 \in P^*$ we must have $q \cdot p_1 \leq q \cdot p_2$. So P is partially ordered as a free path semigroup by $\leq$. $\square$

Let P be a free path semigroup generated by $(P_0,\ P_1)$. Let R be a semigroup with identity set E. Consider a semigroup morphism $\pi : P \longrightarrow R$ such that $\pi\big|_{P_0}(P_0) = E$ and for $p_1,\ p_2 \in P^*$,

$$\pi(p_1)R = \pi(p_2)R \Leftrightarrow R\pi(p_1) = R\pi(p_2).$$

Define a relation $\sim$ on P by

$$p_1 \sim p_2 \Leftrightarrow \pi(p_1)R = \pi(p_2)R.$$

Clearly, $\sim$ is reflexive, symmetric, and transitive so $\sim$ is an equivalence relation; for each $p \in P^*$ denote its equivalence class under $\sim$ by $\bar{p}$. Note that if $p_1 \sim p_2$ then $\sigma(p_1) = \sigma(p_2)$ and $\tau(p_1) = \tau(p_2)$. It follows every suitable order on P determines a minimal representative from the equivalence class $\bar{p}$. Given a suitable order $\leq$ on P, set

$$\mathcal{M} = \{p \in P |\ p \text{ is minimal under} < \text{in } \overline{p}\}.$$

Then $\mathcal{M}$ is a set of minimal representatives of the equivalence classes of P. So for every $p \in {}_iP$, $p \in \mathcal{M}$ if and only if $\pi(p)R = \pi(q)R \Rightarrow p \leq q$ for all $q \in {}_iP$. Then $\mathcal{M}$ has the following property.

Lemma II.8. *The right (left) semigroup ideal generated by Im* π *is generated by* $\pi(\mathcal{M})$*. Furthermore, as a subset of* P*,* $\mathcal{M}$ *is closed under subpaths.*

Proof. The first statement follows from the definition of $\mathcal{M}$.

For the second statement, it suffices to if $p_1 \cdot p_2 \in \mathcal{M}$ then both $p_1,\ p_2 \in \mathcal{M}$. So let $p_1 \cdot p_2 \in \mathcal{M}$ and assume $p_1 \notin \mathcal{M}$. Then there exists $q_1 \in P^*$ such that $p_1 \sim q_1$ and $q_1 < p_1$. By properties of suitable orders $q_1 < p_1 \Rightarrow q_1 \cdot p_2 < p_1 \cdot p_2$ and

$$R\pi(p_1) = R\pi(q_1) \Rightarrow R\pi(p_1)\pi(p_2) = R\pi(p_1 \cdot p_2) = R\pi(q_1 \cdot p_2) = R\pi(q_1)\pi(p_2).$$

So $q_1 \cdot p_2 \sim p_1 \cdot p_2$ and this implies $p_1 \cdot p_2 \notin \mathcal{M}$, a contradiction. Similarly, we must have $p_2 \in \mathcal{M}$. $\square$

Given a free path semigroup P with suitable order $<$, and semigroup morphism $\pi : P \longrightarrow R$ as above, we define

$$\Gamma^2 = \{p \in P \setminus \mathcal{M} | \ p = p_1 \cdot p_2 \text{ with } p_1, p_2 \in P \setminus P_0, \text{ then } p_1, \ p_2 \in \mathcal{M}\}.$$

The set Γ^2 consists of those paths not in $\mathcal{M}$, but whose every proper subpath is in $\mathcal{M}$. Let $\Gamma^0 = P_0$ and $\Gamma^1 = P_1$. Extending our previous notation, we put ${}_i\mathcal{M} = \mathcal{M} \cap {}_iP$ and ${}_i\Gamma^k = \Gamma^k \cap {}_iP$.

Once the set Γ^2 is known for a given semigroup morphism $\pi : P \longrightarrow R$, we may recursively define additional subsets of P, which will be denoted by $\Gamma^k, \ k \geq 3$.

Definition II.9. The set $\Gamma^{k+1} \subseteq P^*$ will be defined as follows.

The path p is in Γ^{k+1} if

(a) $p = \gamma \cdot \alpha$ with $\gamma \in \Gamma^k$ and $\alpha \in \mathcal{M} \setminus \Gamma^0$.

(b) $p = \gamma \cdot \alpha_1 \cdot \alpha_2$, with $\alpha_1, \alpha_2 \in \mathcal{M}$, $\gamma \cdot \alpha_1 \in \Gamma^k$ and $\gamma \in \Gamma^{k-1}$, then $\alpha_1 \cdot \alpha_2 \in P^* \setminus \mathcal{M}$.

(c) p does not equal $\gamma \cdot \beta$, where $\beta \in P^* \setminus P_0$ and γ satisfies (a) and (b) above.

Now a word about notation. According to our established convention, we write general elements of a path semigroup P using the letters p and q. Given such a path p, we may need to describe (if possible) how p relates to elements of Γ^k. We shall reserve the letter γ, decorated with sub and superscripts, to denote elements of Γ^k for $k \geq 0$. In particular, if $p \in \Gamma^k$, then we write $p = \gamma_p^k$; the superscript denotes containment in the set Γ^k and the subscript will prove useful later on. In general, a path p need not be contained in a given Γ^k. However, it may be possible to factor p with a left factor in some Γ^k. In this case, we will use the letter β, decorated with sub and superscripts, to describe the right factor. So $p = \gamma_p^k \cdot \beta_p^k$ is a decomposition of p with left factor $\gamma_p^k \in \Gamma^k$ and $\beta_p^k \in P$. For elements of $\mathcal{M}$, we reserve the letter α. In some instances, the β_p^k above may actually belong to $\mathcal{M}$. If this happens, we write α_p^k instead. Since $\Gamma^0 \cup \Gamma^1 \subseteq \mathcal{M}$, the notation we've chosen is not always consistent, but we believe this inconsistency will be resolved by context. With several applications, this notation becomes cumbersome. In the event of such applications, we may define new paths in order to spare the use of subscripts and superscripts.

A path $p \in P^*$ need not be an element of any Γ^k, $k \geq 0$. If p satisfies (a) and (b), but not (c) of Definition II.9 we have the following key property, stated without proof in [4].

Lemma II.10. *For $k \geq 0$, if $p = \gamma_p^k \cdot \alpha_p^k$, with $q = \gamma_p^k$ and $q = \gamma_q^{k-1} \cdot \alpha_q^{k-1}$, implies $\alpha_q^{k-1} \cdot \alpha_p^k \in P^* \setminus \mathcal{M}$, then p may be uniquely factored as $p = \gamma_p^{k+1} \cdot \alpha_p^{k+1}$ with $\gamma_p^{k+1} \in \Gamma^{k+1}$.*

Proof. First, we must show p has a left factor that satisfies Definition II.9 as an element of Γ^{k+1}. By hypothesis, p satisfies parts (a) and (b) of Definition II.9. Choose α a left factor of α_p^k minimal under length (number of arrows) such that $\alpha_q^{k-1} \cdot \alpha \notin \mathcal{M}$. Such an α must exist since $\alpha_q^{k-1} \cdot \alpha_p^k \notin \mathcal{M}$. We claim $\bar{p} = \gamma_q^{k-1} \cdot \alpha_q^{k-1} \cdot \alpha = \gamma_p^k \cdot \alpha$ is an element of Γ^{k+1}. Since $\mathcal{M}$ is closed under subpaths, $\alpha \in \mathcal{M}$ and $\bar{p}$ satisfies parts (a) and (b) of Definition II.9. Suppose $\bar{p} = \hat{p} \cdot \beta$ with $\hat{p} \in \Gamma^{k+1}$. Since γ_p^k is a subpath of $\bar{p}$, it must also be a subpath of $\hat{p} \cdot \beta$. Factor $\hat{p}$ as $\gamma^k_{\hat{p}} \cdot \alpha^k_{\hat{p}}$. If $\gamma^k_p \neq \gamma^k_{\hat{p}}$, then one is a subpath of the other, and we violate part (c) of Definition II.9 for Γ^k. So $\gamma^k_p = \gamma^k_{\hat{p}}$. If we set $\hat{q} = \gamma^k_{\hat{p}}$ then we must have $\gamma^{k-1}_q = \gamma^{k-1}_{\hat{q}}$ for the same reason. Since $\bar{p} = \gamma^k_p \cdot \alpha = \gamma^k_p \cdot \alpha^k_{\hat{p}} \cdot \beta$, we must have $\alpha^k_{\hat{p}}$ a subpath of α and hence a subpath of α^k_p. But $\gamma^{k-1}_{\hat{q}} \cdot \alpha^k_{\hat{p}} = \gamma^{k-1}_q \cdot \alpha^k_{\hat{p}} \notin \mathcal{M}$, which by our choice of α, implies α is a subpath of $\alpha^k_{\hat{p}}$. Hence $\alpha = \alpha^k_{\hat{p}}$, $\beta = v_{\tau(\hat{p})}$ and $\bar{p}$ satisfies Definition II.9. So $\bar{p} = \gamma^{k+1}_p$ and $p = \gamma^{k+1}_p \cdot \alpha^{k+1}_p$, where $\alpha \cdot \alpha^{k+1}_p = \alpha^k_p$.

If p also factors as $\gamma^{k+1} \cdot \alpha^{k+1}$ with $\gamma^{k+1} \in \Gamma^{k+1}$, then $\gamma^{k+1}_p = \gamma^{k+1}$, otherwise one is a subpath of the other, violating part (c) of Definition II.9 for Γ^{k+1}. It follows $\alpha^{k+1}_p = \alpha^{k+1}$ and the factorization for p is unique. $\square$

CHAPTER III

A CLASS OF BINOMIAL RINGS

We now turn to a certain class of artinian rings called binomial rings (see [10] and [11]). For the remainder of this chapter, R will be an indecomposable ring with identity. Using conventional notation, we let $J = J(R)$ denote the Jacobson radical of R. Recall that R is *semiperfect* if R/J is semisimple and idempotents lift modulo $J(R)$. If R is artinian then by Hopkins Theorem, J is a nilpotent ideal and R/J is a semisimple ring.

III.1. Cleft Rings

Definition III.1. A semiperfect ring R is *cleft* if it contains a subring $S \cong R/J(R)$ such that $R = S \oplus J(R)$ as abelian groups. (Hence, also as S-modules.)

Examples of cleft rings are seen throughout algebra. Any finite-dimensional algebra over a field K is a cleft ring as are group rings, graded rings, quotients of path algebras, and square-free rings.

In Lemma 10 of [12], the author showed an artinian ring R is cleft if only if e_iRe_i is cleft for each basic idempotent e_i. In fact, the cleft property is preserved under Morita equivalence (see Theorem III.3).

We will assume R is a cleft semiperfect ring with subring S and complete set $E = \{e_i |\ i \in I\} \subset S$. Note that for any finitely generated projective left R-module P, there is an indexed set of idempotents D from E that gives the following R-module and abelian group decompositions

$$ {}_R P \cong \bigoplus_{e_i \in D} Re_i \qquad\qquad {}_{\mathbb{Z}} P \cong \bigoplus_{e_i \in D} Re_i = \left(\bigoplus_{e_i \in D} Se_i \right) \oplus \left(\bigoplus_{e_i \in D} Je_i \right). $$

Since isomorphism clearly preserve the cleft property, we will assume the above isomorphisms are actually equalities.

Let $\bar{f} \in \mathrm{End}_R(P/JP)$. Since ${}_R P$ is projective, $\bar{f}$ lifts to a map $f \in \mathrm{End}_R(P)$. However, this lifting need not be unique.

Consider $\bar{f}(e_i + Je_i)$ for $e_i \in D$. Let $s_i = e_i s_i e_i \in e_i S e_i$ such that $\bar{f}(e_i + J) = s_i + Je_i$. If $s_i + Je_i = t_i + Je_i$ for some other $t_i \in e_i S e_i$, then we have $s_i - t_i \in Je_i$. Because S is a ring, S is closed under addition. Hence $s_i - t_i = 0$ and $s_i = t_i$.

Thus, for each $e_i \in D$ and $\bar{f} \in \mathrm{End}_R(P/JP)$, there exists a unique $\hat{f}(e_i) \in e_i S e_i$ such that $\bar{f}(e_i + Je_i) = \hat{f}(e_i) + Je_i$.

Define $\Phi : \mathrm{End}_R(P/JP) \longrightarrow \mathrm{End}_R(P)$ by

$$ \Phi(\bar{f})(e_i r) = \hat{f}(e_i) r $$

for all $r \in R$ and $e_i \in D$. Then we have the following.

Lemma III.2. *The map $\Phi : End_R(P/JP) \longrightarrow End_R(P)$ sending $\bar{f}$ to f is a ring monomorphism.*

Proof. If $\Phi(\bar{f}) = 0$, then for all $e_i \in D$, $\bar{f}(e_i + Je_i) = 0$ showing $\bar{f} = 0$ and Φ is injective.

Let $\Phi(\bar{f}_1),\ \Phi(\bar{f}_2) \in \text{Im } \Phi$. Note that

$$(\bar{f}_1 + \bar{f}_2)(e_i + Je_i) = \hat{f}_1(e_i) + \hat{f}_2(e_i) + Je_i$$

and

$$\bar{f}_1\bar{f}_2(e_i + Je_i) = \bar{f}_1(e_i + Je_i) \cdot \hat{f}_2(e_i) = \hat{f}_1(e_i)\hat{f}_2(e_i) + Je_i.$$

Both $\hat{f}_1(e_i) + \hat{f}_2(e_i)$ and $\hat{f}_1(e_i)\hat{f}_2(e_i)$ are elements of e_iSe_i. Hence, by definition of Φ, $\Phi(\bar{f}_1 + \bar{f}_2) = \Phi(\bar{f}_1) + \Phi(\bar{f}_2)$, and $\Phi(\bar{f}_1\bar{f}_2) = \Phi(\bar{f}_1)\Phi(\bar{f}_2)$. Finally, we have $1_{P/JP} = e_1 + \ldots + e_n + JP \longrightarrow e_1 + \ldots + e_n = 1_P$ under Φ, so Im Φ is a subring of $\text{End}_R(P)$ with Im $\Phi \cong \text{End}_R(P/JP)$. □

This result leads to the following.

Theorem III.3. *Let $R \approx \hat{R}$. If R is cleft with subring S, then $\hat{R}$ is cleft with subring $\hat{S}$ and $S \approx \hat{S}$.*

Proof. Since $R \approx \hat{R}$, $\hat{R} \cong \text{End}_R(P)$ for some progenerator ${}_RP$. Since the cleft property is preserved under isomorphisms, we may assume $\hat{R} = \text{End}_R(P)$ with the module $P = \bigoplus_{e_i \in D} Re_i$ for some indexed set of idempotents D from E. Set

$$\hat{S} = \{\Phi(\bar{f})|\ \bar{f} \in \text{End}_R(P/JP)\}.$$

By lemma III.2, $\hat{S} \cong \text{End}_R(P/JP)$. Since R is cleft, $R/J \cong S$ as rings and

$$\text{End}_R(P/JP) = \text{End}_{R/J}(P/JP) \cong \text{End}_S(P/JP).$$

Note that ${}_SP/JP$ is a progenator (see exercise 21.10 of [3]). By Corollary 22.4 of [3], $S \approx \mathrm{End}_S(P/JP) \cong \hat{S}$. By Corollary 17.12 of [3], $\mathrm{End}_R(P)/J(\mathrm{End}_R(P)) \cong \mathrm{End}_R(P/JP) \cong \hat{S}$. By definition of $\{\Phi(\bar{f})|\ \bar{f} \in \mathrm{End}_R(P/JP)\}$ and Corollary 17.12 of [3],

$$\hat{S} \cap J(\mathrm{End}_R(P)) = \{\Phi(\bar{f})|\ \bar{f} \in \mathrm{End}_R(P/JP)\} \cap \mathrm{Hom}_R(P,\ JP) = 0.$$

So $\hat{R}$ is cleft, with subring $\hat{S} \approx S$. □

A consequence of the above lemma, we get a generalization of the result by Vinograde [12].

Corollary III.4. *A semiperfect ring R is cleft if and only if its basic ring is cleft.*

Proof. By Theorem 27.14 of [3], a semiperfect ring R is Morita equivalent to its basic ring $\hat{R}$. But by Theorem III.3, R is cleft if and only if $\hat{R}$ is cleft. □

Unless specified otherwise, from now on we assume that a cleft ring R is a basic ring with basic set $E \subset S$.

III.2. Binomial Rings

Let R be a basic right artinian ring with basic set $E = \{e_1,\ e_2, \ldots, e_n\}$. The *right quiver* of R is a directed graph Γ with vertex E and h_{ij} arrows

$$e_j \xrightarrow[\alpha_k]{} e_i \qquad\qquad k = 1, \ldots, h_{ij},\ 1 \leq i,\ j \leq n$$

where h_{ij} is the multiplicity of e_iR/e_iJ in e_jJ/e_jJ^2. The directed graph Γ gives rise to an associated path semigroup Q with vertex set $Q_0 = \{v_1,\ v_2, \ldots, v_n\}$ and arrows $Q_1 = \{\alpha_k |\ k = 1, \ldots, h_{ij},\ 1 \leq i,\ j \leq n\}$. Recall the set maps $\sigma_Q : Q \longrightarrow Q_0$ and $\tau_Q : Q \longrightarrow Q_0$ such that $q = \sigma(q) \cdot q \cdot \tau(q)$ for all $q \in Q^*$. Denote by P the free path semigroup generated by $(Q_0,\ Q_1)$ subject to σ_Q and τ_Q. Choose $R_1 \subseteq \cup_{i,j=1}^{n} e_iRe_j$ such that

$$J/J^2 = \bigoplus_{x \in R_1} (xR + J^2)/J^2$$

is a semisimple decomposition of J/J^2 as simple right R-modules. We choose a bijection

$$\pi = \pi_{E,R_1} : P_0 \cup P_1 \to E \cup R_1$$

with

$$\pi(p \cdot q) = \pi(p)\pi(q)$$

for all $p, q \in P_0 \cup P_1 \subset P^*$ with $p \cdot q \in P_0 \cup P_1$. It follows $\pi(P_0) = E$ and $\pi(P_1) = R_1$. Extend the map from P^* to R by defining

$$\pi(a_1 \cdot a_2 \cdots a_k) = \pi(a_1)\pi(a_2) \cdots \pi(a_k).$$

Our interest is in exploring the properties of cleft binomial rings. In general, binomial rings need not be cleft. We begin with some basic lemmas needed to describe the structure of a cleft ring.

Lemma III.5. *Let R be a basic left artinian ring, and let E, Γ, P, R_1 and $\pi : P^* \to R$ be as above. Let $p \in P^*$ with $\pi(p) \neq 0$, and let $e, f, \in E$ and $i \geq 0$ such that*

$\pi(p) = e\pi(p)f \in J^i \setminus J^{i+1}$. *Then, as simple right R-modules, we have*

$$(\pi(p)R + J^{i+1})/J^{i+1} \cong \pi(p)R/\pi(p)J \cong eR/eJ.$$

Proof. See Lemma 2.1 of [11]. □

If for each $i \geq 0$ there is a $p \in P^*$ with $0 \neq \pi(p) \in J^i \setminus J^{i+1}$ then we can find a subset T_i of $\pi(P^*) \setminus \{0\}$ such that

$$J^i/J^{i+1} = \bigoplus_{t \in T_i} (tR + J^{i+1})/J^{i+1}.$$

Take $T^* = \cup T_i$. We sometimes refer to $(\pi,\ T^*)$ as a right system of R. If R is left artinian, then a left system is defined similarly. Note that a ring R with a left system need not have a right system (see example Example 1.1 of [6]). Left and right systems can be defined using different basic sets $E = T_0$ and $\hat{E} = \hat{T}_0$. Therefore, a ring R with both left and right systems need not have equality between the two systems (see the Definition III.6 below). The following was first explored by Sklar in [11].

Definition III.6. Let R, E, J, Γ, P, R_1 and $\pi : P^* \to R$ be as before. Now let T^* be a subset of $\pi(P^*) \setminus \{0\}$, and let $T_i = \{t \in T^* | t \in J^i \setminus J^{i+1}\}$, for each $i \geq 0$. Then (π, T^*) is a *binomial system* for R if the following two conditions hold:

(B.1) $J^i/J^{i+1} = \bigoplus_{t \in T_i} (Rt + J^{i+1})/J^{i+1} = \bigoplus_{t \in T_i} (tR + J^{i+1})/J^{i+1}$

(B.2) For every $p \in P^*$ with $\pi(p) \neq 0,$ there exists an element $t_p \in T^*$ with $R\pi(p) = Rt_p$ and $\pi(p)R = t_pR$.

We say R is a *binomial ring* if there exists a binomial system $(\pi,\ T^*)$ for R, for some choice of E and R_1. As noted in [11], given a binomial system $(\pi,\ T^*)$ the set $T = T^* \cup \{0\}$ is an algebra semigroup under the operation $\cdot$ where $t_1 \cdot t_2 = t_3 \in T$ such that $Rt_1t_2 = Rt_3$.

Given an artinian ring R and finitely generated R-module M, we denote by $c(M)$ the composition length of M. For a left module ${}_RM$, denote by $c_i(M)$ the multiplicity of Re_i/Je_i as a factor (up to ismorphism) in a composition series for ${}_RM$. Similarly, for a right module M_R denote by $d_i(M)$ the multiplicity of e_iR/e_iJ as a factor in a composition series for M_R. Note for all $t \in T$ we have $t = e_ite_j$ for some unique idempotents $e_i,\ e_j \in E$. Then for any binomial ring with binomial system $(\pi,\ T^*)$,

$$c_i(Je_j/J^2e_j) = |e_iT_1e_j| = d_j(e_iJ/e_iJ^2).$$

Hence, the left quiver of a binomial ring R is the order theoretic dual of the right quiver Γ.

III.3. Locally Square-Free Rings

We now briefly discuss a several types of binomial rings. For a basic artinian ring R, each finitely generated left R-module M has a composition series.

Definition III.7. A finitely generated indecomposable R-module M is *square-free* if

$$c_i(M) \leq 1$$

for all $i = 1, 2, \ldots, n$.

A ring is *left (right) square-free* if each of its indecomposable projective left (right) modules is square-free. We say a ring R is *square-free* if R is both left and right square-free. By Theorem 3.6 of [11], every basic square-free ring is binomial.

We can extend this definition to a more "local" version. Recall the Loewy length of a module $l = l(M)$ is the smallest positive integer such that $J^l M = 0$.

Definition III.8. A finitely generated indecomposable left R-module M with Loewy length l is *locally square-free* if

$$c_i(J^k M / J^{k+1} M) \leq 1$$

for all $i = 1, 2, \ldots, n$ and $k = 0, 1, \ldots, l-1$. Locally square-free is defined similarly for a right R-module.

Definition III.9. A ring R is *left locally square-free* if the left regular module ${}_RR = \oplus_{i=1}^n Re_i$ is the direct sum of locally square-free indecomposable projective modules; similarly, R is *right locally square-free* if the right regular module $R_R = \oplus_{i=1}^n e_iR$ is the direct sum of locally square-free indecomposable projective modules. A ring R is *locally square-free* if it is both left locally square-free and right locally square-free.

Note the property of being locally square-free is a Morita invariant, so it is sufficient to restrict our study of locally square-free rings to basic locally square-free rings.

If R is left locally square-free, then $c_i(Je_j/J^2e_j) \leq 1$ for all i and j, implying the right quiver $\Gamma = \Gamma(R)$ has at most one (directed) arrow between any two vertices. If R is right locally square-free, a similar result holds for the left quiver.

All square-free rings are locally square-free, but a locally square-free ring need not be square-free. For example, $\mathbb{Z}_4$ is locally square-free but not square-free. Square-free and locally square-free rings have many properties in common. The following two lemmas are easy extensions of results found in [6].

Lemma III.10. *Let R be an artinian ring and let ${}_RM$ be an indecomposable module. If $e_i \in R$ is a primitive idempotent, then $c_i(M) = 1$ if and only if $Re_ix = Re_iM$ for each $x \in M$ with $e_ix \neq 0$.* $\square$

Lemma III.11. *Let R be a basic (left) locally square-free and let $e_i, e_j \in R$ be primitive idempotents. If $0 \neq e_ire_j + J^{k+1}$ is an element of J^k/J^{k+1} for some $r \in R$ and $k \geq 0$, then*

$$R/J \cdot (e_ire_j + J^{k+1}) = Re_iJ^ke_j + J^{k+1} \leq J^k/J^{k+1}. \qquad \square$$

Since R is a basic artinian ring with basic set $E = \{e_1, \ldots, e_n\}$ we see that

$$R/J \cong e_1Re_1/e_1Je_1 \dotplus \ldots \dotplus e_nRe_n/e_nJe_n = \overline{e_1Re_1} \dotplus \ldots \dotplus \overline{e_nRe_n}$$

as rings. Each $\overline{e_iRe_i}$ is a division ring (Corollary 17.20 of [3]).

Lemma III.12. *Let R be an indecomposable basic locally square-free ring with basic set $E = \{e_i|\ i \in I\}$. Then*

$$dim_{\overline{e_iRe_i}}((e_iJ^ke_j + J^{k+1})/J^{k+1}) = dim_{\overline{e_jRe_j}}((e_iJ^ke_j + J^{k+1})/J^{k+1}) \leq 1$$

for all $e_i, e_j \in E$ and all k.

Proof. Let $t \in J^k$ so that $0 \neq e_ite_j + J^{k+1} \in (e_iJ^ke_j + J^{k+1})/J^{k+1}$.

From lemma III.11

$$\begin{aligned} \overline{e_iRe_i} \cdot (e_ite_j + J^{k+1}) &= (e_iRe_ite_j + J^{k+1}) \\ &= e_iRe_iJ^ke_j + J^{k+1} \\ &= e_iRe_iRJ^ke_j + J^{k+1} = e_iJ^ke_j + J^{k+1} \end{aligned}$$

(the last equality follows from the fact that e_iR is an idempotent right ideal).

Hence $t + J^{k+1}$ is a basis for $(e_iJ^ke_j + J^{k+1})/J^{k+1}$ over $\overline{e_iRe_i}$. (note a similar result holds for $(e_iJ^ke_j + J^{k+1})/J^{k+1}$ as a right $\overline{e_jRe_j}$-module) □

The following fact will be used in a later theorem.

Lemma III.13. *If R is a basic locally square-free ring with radical $J = J(R)$ then $c(_R(J^i/J^{i+1})) = c((J^i/J^{i+1})_R)$ for all i.*

Proof. Note that $c(M) = \sum_{k=1}^{n} c_k(M)$ for any module M of finite length. Then

$$\begin{aligned} c({}_R J^i/J^{i+1}) = \sum_{k=1}^{n} c_k({}_R J^i/J^{i+1}) &= \sum_{k=1}^{n} c({}_{e_k R e_k} e_k J^i/J^{i+1}) \\ &= \sum_{k=1}^{n} c({}_{\overline{e_k R e_k}} e_k J^i/J^{i+1}) \\ &= \sum_{k=1}^{n} \dim_{\overline{e_k R e_k}} (e_k J^i/J^{i+1}). \end{aligned}$$

Similarly, $c(J^i/J^{i+1}_R) = \sum_{k=1}^{n} \dim(J^i e_k/J^{i+1})_{\overline{e_k R e_k}}$.

But R is locally square-free, so by Lemma III.12,

$$\begin{aligned} c({}_R(J^i/J^{i+1})) &= \sum_{k=1}^{n} \dim_{\overline{e_k R e_k}} (e_k J^i/J^{i+1}) \\ &= \sum_{k=1}^{n} \dim(J^i e_k/J^{i+1})_{\overline{e_k R e_k}} = c(((J^i/J^{i+1})_R). \end{aligned}$$

□

We claim that every basic locally square-free ring is a binomial ring. First we have the following lemma:

Lemma III.14. *Let R be an artinian ring with radical J, and let M be a locally square-free left R-module. Suppose there are submodules K, N of M such that*

$$K \subseteq J^{l-1}M, N \subseteq J^{l-1}M \text{ and } K \nsubseteq J^l M, N \nsubseteq J^l M \text{ for some } l,$$

with $(K + J^l M)/J^l M$ and $(N + J^l M)/J^l M$ isomorphic to the same simple left R-module. Then $K = N$.

Proof. Let $\overline{Re_i} \cong (K + J^\ell M)/J^\ell M$ and note that $K + N \subseteq J^{\ell-1}M$. Then

$$c_i((K + N + J^\ell M)/J^\ell M) + c_i(K \cap N + J^\ell M)/J^\ell M)$$

$$= c_i((K + J^\ell M)/J^\ell M) + c_i((N + J^\ell M)/J^\ell M).$$

Since M is locally square-free, each $J^{\ell-1}M/J^\ell M$ is square-free $R/J-$module. Hence

$$c_i((K + N + J^\ell M)/J^\ell M) = c_i((K + J^\ell M)/J^\ell M) = c_i((N + J^\ell M)/J^\ell M) = 1$$

implying that $c_i(K \cap N + J^\ell M)/J^\ell M) = 1$.

Now $K \cap N$ is a submodule of both K and N. If $K \cap N \subseteq JK \cup JN$, then

$$(K \cap N + J^\ell M)/J^\ell M) \subseteq (JK \cup JN + J^\ell M)/J^\ell M) \subseteq J \cdot J^{\ell-1}M/J^\ell M = 0$$

contradicting that $c_i(K \cap N + J^\ell M)/J^\ell M) = 1$. So $K \cap N \nsubseteq JK \cup JN$ implying $K \cap N \nsubseteq JK$ and $K \cap N \nsubseteq JN$. But $JK \subset K$, $JN \subset N$ are unique maximal submodules; hence $K = K \cap N = N$. □

We now have the following

Theorem III.15. *Every basic locally square-free ring is binomial.*

Proof. We must show that a basic locally square-free ring satisfies properties (B.1) and (B.2) of Definition III.6. Let R be a basic locally square-free ring with E, J, Γ, P, R_1 and $\pi : P^* \longrightarrow R$ defined as before. Each $(R\pi(p) + J^{k+1})/J^{k+1}$ is a simple left R-module whenever $p \in P^*$ and $0 \neq \pi(p) \in J^k/J^{k+1}$ $(k \geq 0)$

So we can find a subset T^* of $\pi(P^*) \setminus \{0\}$ such that for each $k \geq 0$

$$J^k/J^{k+1} = \bigoplus_{t \in T_k} (Rt + J^{k+1})/J^{k+1}$$

where $T_k = \{t \in T^* | t \in J^k \setminus J^{k+1}\}$. We must show that this T^* also gives a decomposition of J^k/J^{k+1} as a right R-module as well as show that this choice of T^* satisfies property (B.2) of Definition III.6.

Let

$$L_k = \sum_{t \in T_k} (tR + J^{k+1})/J^{k+1} = \bigoplus_{e_i \in E} \sum_{t = e_i t \in T_k} (tR + J^{k+1})/J^{k+1}.$$

Since R is locally square-free, we know for all $e_j \in E$, Re_j is a locally square-free left R-module. So for every pair of idempotents $e_i, e_j \in E$

$$|e_i T_k e_j \setminus \{0\}| = c_i((J^k e_j + J^{k+1})/J^{k+1}) \leq 1.$$

For every $e_i \in E$, and every pair of elements $t \neq t'$ in $e_i T_k \setminus \{0\}$, we have $t = te_j$ and $t' = t'e_j'$ for some $e_j \neq e_j' \in E$.

So

$$(tR + J^{k+1})/J^{k+1} \cong e_j R/e_j J \ncong e_j' R/e_j' J \cong (t'R + J^{k+1})/J^{k+1}.$$

Therefore

$$L_k = \bigoplus_{e_i \in E} \bigoplus_{t = e_i t \in T_k} (tR + J^{k+1})/J^{k+1} = \bigoplus_{t \in T_k} (tR + J^{k+1})/J^{k+1} \leq J^k/J^{k+1}.$$

By Lemma III.13

$$c(J^k/J^{k+1}_R) = c({}_R J^k/J^{k+1}) = |T_k| = c(L_k).$$

So $L_k = J^k/J^{k+1}$ and property (B.1) holds for $(\pi,\ T^*)$. Let $p \in P^*$ such that $\pi(p) \neq 0$ and let $e_i, e_j \in E$ and $i \geq 0$ such that $\pi(p) = e_i \pi(p) e_j \in J^k/J^{k+1}$. Note that $R\pi(p) \subset J^k$ and $R\pi(p) \nsubseteq J^{k+1}$ with

$$(R\pi(p) + J^{k+1})/J^{k+1} \cong R\pi(p)/J\pi(p) \cong \overline{Re_i}.$$

Then $(R\pi(p)+J^{k+1})/J^{k+1} \cong (Rt+J^{k+1})/J^{k+1}$ for some $t = e_i t e_j \in T_k$. This choice of t is unique, since $|e_i T_k e_j \setminus \{0\}| \leq 1$.

Now Re_j is a locally square-free module with $Rt \subseteq J^k e_j$, $R\pi(p) \subseteq J^k e_j$, $Rt \nsubseteq J^{k+1}e_j$, $R\pi(p) \nsubseteq J^{k+1}e_j$
and $(R\pi(p)+J^{k+1})/J^{k+1} \cong (Rt+J^{k+1})/J^{k+1}$. By Lemma III.14, $Rt = R\pi(p)$. Finally, by uniqueness of t, we must also have $tR = \pi(p)R$. Therefore property (B.2) holds for $(\pi,\ T^*)$ and R is a binomial ring.

□

III.4. Cleft Binomial Rings

We now explore binomial rings that also have the cleft property. Note that a binomial ring need not be cleft, although square-free rings are examples of binomial rings that are automatically cleft (see Theorem 1.4 of [6]). However, if in addition to being cleft with subring S, a ring R is also binomial, then condition **(B.1)** is equivalent to

$$J^i/J^{i+1} = \bigoplus_{t\in T_i}(St+J^{i+1})/J^{i+1} = \bigoplus_{t\in T_i}(tS+J^{i+1})/J^{i+1}.$$

Note that locally square-free cleft rings (including square-free rings) with binomial system $(\pi,\ T^*)$ have the property that, for all $t \in T_k$,

$$(St+J^{k+1})/J^{k+1} = (tS+J^{k+1})/J^{k+1}$$

as sets. In particular, for all $s \in S, t \in T_k$ and $k \geq 0$, $st + J^{k+1} = t\hat{s} + J^{k+1}$ for some $\hat{s} \in S$. This need not be the case for general binomial rings.

Definition III.16. A binomial system $(\pi,\ T^*)$ for a cleft binomial ring with subring S is *strong* if for all $s \in S$, $t \in T_k$ and $k \geq 0$,

$$st + J^{k+1} = t\hat{s} + J^{k+1}$$

for some $\hat{s} \in S$.

Clearly, every binomial system for a cleft locally square-free ring is strong. Given a binomial ring R with strong binomial system and $t = e_i t e_j \in T_k$ for $e_i, e_j \in E$, there is an isomorphsim of division rings $\phi : \overline{e_j R e_j} \to \overline{e_i R e_i}$ defined by:

$$tr + J^{k+1} = \phi(r)t = J^{k+1}.$$

This gives $(Rt + J^{k+1})/J^{k+1}$ a right $\overline{e_j R e_j}$-module structure, defined by

$$(r't + J^{k+1})e_j r e_j = r'\phi(e_j r e_j)t + J^{k+1}.$$

If R is assumed to be indecomposable, the right quiver of R, $\Gamma(R)$ is connected. Then for any pair of primitive idempotents $e_i,\ e_j \in E$, we have a sequence of primitive idempotents $e_i = f_1, f_2, \ldots, f_k = e_j$ and a sequence $t_i = t_1,\ t_2, \ldots, t_k \in T_1$ such that $t_\ell = f_\ell t_\ell f_{\ell+1}$. This gives isomorphisms between all pairs of division rings $\overline{e_i R e_i},\ \overline{e_j R e_j},\ e_i,\ e_j \in E$ (see Chapter 7 of [3]).

In our remaining discussion of cleft binomial rings, we will assume all binomial systems are strong. Indeed, unless stated otherwise R is an indecomposable basic cleft binomial ring with strong binomial system.

We have $\overline{e_iRe_i} \cong e_iSe_i$ for all $e_i \in E$, so we may identify $\overline{e_iRe_i}$ with e_iSe_i. By assumption $(\pi,\ T^*)$ is strong and $e_iSe_i \cong e_jSe_j$ for all $e_i,\ e_j \in E$.

If R is binomial with system $(\pi,\ T^*)$, let $\ell_k = |T_k|$. The cleft property gives a nice way to write elements of R.

Lemma III.17. *Let R, E, Γ, P, R_1 and $\pi : P^* \longrightarrow R$ be as before. Suppose R has Loewy length $L(R) = m$. If R is cleft with subring $S \cong R/J$, then for all $r \in R$*

$$r = \sum_{i=1}^{m} \sum_{j=1}^{\ell_i} t_{i,j} s_{i,j}$$

where $s_{i,j} \in S$ and $t_{i,j} \in T_i$.

Proof. By induction on Loewy length $L(R) = m + 1$. If $m = 1$, then

$$R = R/J = \bigoplus_{t \in T_i} (tR + J)/J = \bigoplus_{t \in T_1} tS$$

since $R = S$ when $m = 1$. So elements of R can be written in the desired form. Now assume $L(R) = 2$ and R has binomial system $(\pi,\ T^*)$ with $T^* = T_0 \cup T_1$. Note that $T_0 = E$. Also

$$J = J/J^2 = \bigoplus_{t \in T_1} (tR + J^2)/J^2 = \bigoplus_{t \in T_1} tS.$$

So $x \in J$ has the form $x = \sum_{j=1}^{\ell_1} t_{1,j} s_{1,j}$ for $t_{1,j} \in T_1$ and $s_{1,j} \in S$. Since R is cleft, $R = S \bigoplus J$ and $S = \bigcup_{e_i \in E} e_i S$. Then elements of S can be written $s = \sum_{j=1}^{\ell_0} t_{0,j} s_{0,j}$ with $t_{0,j} \in T_0 = E$ and $s_{0,j} \in S$. So any $r \in R$ can be written

$$r = \sum_{j=1}^{\ell_0} t_{0,j} s_{0,j} + \sum_{j=1}^{\ell_1} t_{1,j} s_{1,j} = \sum_{i=0}^{1} \sum_{j=1}^{\ell_i} t_{i,j} s_{i,j}.$$

Assume the result for $2 \leq k \leq m$ and suppose R is a binomial ring with quiver Γ, associated free path semigroup P, system $(\pi,\ T^*)$, and Loewy length $m+1$. Consider the ring R/J^m. Note that R/J^m is cleft, with subring

$$(S + J^m)/J^m \cong S$$

Since $(e_jJ/e_jJ^m)/(e_jJ^2/e_jJ^m) \cong e_jJ/e_jJ^2$, the right quiver of R/J^m is Γ and the associated free path semigroup is P. Then R/J^m is a binomial ring with system $(\bar{\pi},\ U^*)$ where $\bar{\pi}(p) = \pi(p) + J^m$ and $U^* = T^* + J^m$. Let $r + J^m \in R/J^m$. Since $L(R/J^m) \leq m$ by induction there exist $s_{i,j} + J^m \in (S + J^m)/J^m$ and $t_{i,j} + J^m \in \bar{\pi}(P^*) = U^*$ such that

$$\begin{aligned} r + J^m &= \sum_{i=1}^{m-1} \sum_{j=1}^{\ell_i} (t_{i,j} + J^m)(s_{i,j} + J^m) \\ &= \sum_{i=1}^{m-1} \sum_{j=1}^{\ell_i} (t_{i,j}s_{i,j} + J^m) \\ &= \sum_{i=1}^{m-1} \sum_{j=1}^{\ell_i} (t_{i,j}s_{i,j}) + J^m. \end{aligned}$$

So $r - \displaystyle\sum_{i=1}^{m-1} \sum_{j=1}^{\ell_i} t_{i,j}s_{i,j} = x \in J^m$. Now since $L(R) = m+1$,

$$J^m = J^m/J^{m+1} = \bigoplus_{t \in T_m} (Rt + J^{m+1})/J^{m+1} = \bigoplus_{t \in T_m} tR = \bigoplus_{t \in T_m} tS.$$

Hence any $x \in J^m$ is of the form $x = \displaystyle\sum_{j=1}^{\ell_m} t_{m,j}s_{m,j}$ with $s_{m,j} \in S$, $t_{m,j} \in T_m$. Then

$$r = \sum_{i=1}^{m-1} \sum_{j=1}^{\ell_i} t_{i,j}s_{i,j} + \sum_{j=1}^{\ell_m} t_{m,j}s_{m,j} = \sum_{i=1}^{m} \sum_{j=1}^{\ell_i} t_{i,j}s_{i,j}.$$

So every element of R may be written in the desired form. $\square$

III.5. Binomial Rings And Suitable Orders

We continue to investigate basic cleft binomial rings with strong binomial systems. For binomial ring R with binomial system $(\pi,\ T^*)$ and basic set $E = T_0$, we have

$$R = \bigcup_{e_i, e_j \in E} e_i R e_j.$$

With the usual multiplication of the ring, we see that R is a multiplicative semigroup with zero element $\theta_R = 0$. There are set maps $\sigma_R, \tau_R : R \longrightarrow T_0$ defined in the usual way.

Taking P to be the free path semigroup generated by $(T_0,\ T_1)$ subject to σ_R and τ_R, we have $\pi : P \longrightarrow R$ a semigroup morphism. As before, we shall impose a suitable order $<$ on P. Since R is a binomial ring, by Lemma 3.1 of [11] the semigroup morphism π has the property that

$$\pi(p_1)R = \pi(p_2)R \Leftrightarrow R\pi(p_1) = R\pi(p_{2)}.$$

As before, we have an equivalence relation $\sim$ on P and we can choose representatives of each class $\overline{p}$ that are minimal under $<$ to get

$$\mathcal{M} = \{p \in P |\ p \text{ is minimal under } < \text{ in } \overline{p}\}.$$

We would like this choice of $\mathcal{M}$ (dependent upon $<$) to be "compatible" with our starting binomial system $(\pi,\ T^*)$.

Definition III.18. Let R be a cleft binomial ring with cleft subring S. A suitable ordering $<$ that *respects a binomial system* $(\pi,\ T^*)$ is a suitable ordering on P^* such that for $t \in T$ with $t = \pi(p)$

$$\pi(p)R = \pi(q)R \Rightarrow p \leq q.$$

By Lemma 3.1 of [11], the above definition is left-right symmetric. That is, we can replace $\pi(p)R$ and $\pi(q)R$ above with $R\pi(p)$ and $R\pi(q)$.

Given a binomial ring R with $(\pi,\ T^*)$, we may define a suitable ordering $<$ on the free path semigroup P^*. This may or may not be a suitable ordering that respects the binomial system. Define a new binomial system $(\pi,\ U^*)$ for R in the following way. Take

$$\pi : P^* \longrightarrow R \quad \text{as above.}$$

For each $t \in T_k$, let

$$[t] = \{p \in P |\ \pi(p)R = tR\}.$$

Clearly, $\sigma(p) = \sigma(q)$ for all $p,\ q \in P_t$. Since each ${}_iP$ is well-ordered under $<$, we may choose minimal representative from $[t]$. Let $p_t = \min\{p |\ p \in [t]\ \}$ and define

$$U_k = \{\pi(p_t) |\ t \in T_k\}.$$

Take $U^* = \cup U_k$. We shall see that $(\pi,\ U^*)$ is a binomial system for R. In addition, this new system is compatible with $<$ and preserves the strong property of $(\pi,\ T^*)$.

Lemma III.19. *Let R be a binomial ring with binomial system $(\pi,\ T^*)$. Choose a suitable ordering $<$ on P^* and construct the binomial system U^* as above. Then*

$(\pi,\ U^*)$ is a binomial system for R and $<$ is a suitable ordering that respects $(\pi,\ U^*)$. Moreover, if $(\pi,\ T^*)$ is a strong binomial system, then $(\pi,\ U^*)$ is strong as well.

Proof. We begin by showing $(\pi,\ U^*)$ satisfies **(B.1)** and **(B.2)** of Definition III.6. If $p \in P$, then since $(\pi,\ T^*)$ is a binomial system, there exists a $t = \pi(p) \in T^*$ such that $\pi(p)R = tR$ and $R\pi(p) = Rt$. But $tR = uR$ and $Rt = Ru$ for some $u \in U^*$, so **(B.2)** is satisfied. Also for every $t \in T_k$ we have a $u \in U_k$ so that

$$(Rt + J^{k+1})/J^{k+1} = (Ru + J^{k+1})/J^{k+1}$$

and

$$(tR + J^{k+1})/J^{k+1} = (uR + J^{k+1})/J^{k+1}.$$

Since $(\pi,\ T^*)$ is a binomial system we have

$$J^k/J^{k+1} = \bigoplus_{u \in U_k} (Ru + J^{k+1})/J^{k+1}$$

and

$$J^k/J^{k+1} = \bigoplus_{u \in U_k} (uR + J^{k+1})/J^{k+1}.$$

So $(\pi,\ U^*)$ satisfies **(B.1)**. By construction of U^*, $\pi(p)R = \pi(q)R \Rightarrow p \le q$ for all p such that $\pi(p) \in U$. So the suitable order $<$ respects the binomial system $(\pi,\ U^*)$.

Finally, we suppose that $(\pi,\ T^*)$ is a strong binomial system. So for all $s \in S$, $t \in T_k$ and $k \ge 0$,

$$st + J^{k+1} = t\hat{s} + J^{k+1}$$

for some $\hat{s} \in S$. By construction of U, $u = tr = t(s_r + x_r)$ for some $r = s_r + x_r \in R = S \bigoplus J$. Then

$$\begin{aligned} su + J^{k+1} &= st(s_r + x_r) + J^{k+1} \\ &= sts_r + stx_r + J^{k+1} \\ &= t\hat{s}s_r + J^{k+1} = u\bar{s} + J^{k+1} \end{aligned}$$

where the equalities follows since $stx_r \in J^{k+1}$ and $tS + J^{k+1} = uS + J^{k+1}$ implies $t\hat{s}s_r = u\bar{s}$ for some $\bar{s} \in S$. $\square$

Given a suitable ordering $<$ that respects a binomial system $(\pi,\ T^*)$ we now have a useful characterization of the paths used to make T^*. The lemma below follows easily and we state it without proof.

Lemma III.20. *Let R be a cleft binomial ring with subring S and binomial system $(\pi,\ T^*)$. Let $\overline{p}$ denote the equivalence class of p determined by $\pi : P \longrightarrow R$. Let $<$ be a suitable order on P^* that respects the binomial system and denote by $\mathcal{M}$ the set*

$$\mathcal{M} = \{p \in P |\ p \textit{ is minimal under} < \textit{in } \overline{p}\}.$$

If $\mathcal{M}' = \pi^{-1}(T^*)$ is the paths used to make the binomial system, then $\mathcal{M}' = \mathcal{M}$. $\square$

CHAPTER IV

SEMIGROUPS AND RESOLUTIONS

IV.1. Semigroups and Binomial Rings

The purpose of this chapter is to establish the ideas needed to construct projective resolutions using free path semigroups. Throughout this chapter, R will be a basic cleft binomial ring with strong binomial system $(\pi,\ T^*)$. Associated to R is free path semigroup P on the set $P_0 \cup P_1 = \pi^{-1}(T_0 \cup T_1)$ and map

$$\pi : P \longrightarrow R$$

such that $\pi(P)R = R$. Since R is cleft with subring S and $\pi(P_0) = E$, we show that for any subset $X \subseteq P$ of the free path semigroup, we have a right S-module structure, denoted by X_S.

Let $\leq$ denote a suitable order on P and, by Lemma III.19, we may assume $\leq$ is compatible with the the binomial system $(\pi,\ T^*)$. The binomial ring R is a multiplicative semigroup with operation $\cdot$ the usual multiplication. By Lemma III.20 and Lemma II.8 we have a subset of P^*, determined by $\leq$, with $\mathcal{M} = \pi^{-1}(T^*)$ such that $\pi(\mathcal{M})R = R$. In fact, since R is cleft and binomial we have by Lemma III.17 $\pi(\mathcal{M})S = R$. Once $\mathcal{M}$ is known, we may define Γ^k for $k \geq 2$ (with $\Gamma^0 = P_0$ and $\Gamma^1 = P_1$).

IV.2. Module Structure on Subsets of P

We begin by discussing a module structure defined on P. Let S be a basic semisimple ring with basic set $E = \{e_1, e_2, \ldots, e_n\}$. Now

$$S = \bigoplus_{e_i \in E} e_i S e_i$$

and each $e_i S e_i$ is a division ring. Suppose we have a free path semigroup P and semigroup morphism $\pi : P \longrightarrow S$, such that $\pi(P_0) = E$. We introduce a right S-module structure on a subset $X \subseteq P$, denoted by X_S, in the following way. For each $p = p \cdot v_{\tau(p)}$, form the (right) $e_{\tau(p)} S e_{\tau(p)}$ vector space with basis p, denoted by $p e_{\tau(p)} S e_{\tau(p)}$. Then X_S is defined as

$$X_S = \bigoplus_{p \in X} p e_{\tau(p)} S e_{\tau(p)}.$$

The right S-action is given by:

$$s = \sum_{i=1}^{n} e_i s_i e_i \in S \Rightarrow ps = p e_{\tau(p)} s_{\tau(p)} e_{\tau(p)}.$$

One easily checks that this is a well-defined right S-module structure.

We will especially make use of this module structure on the subsets Γ^k and ${}_i\Gamma^k$ of P, for $k \geq 0$. Denoted by Γ^k_S and ${}_i\Gamma^k_S$, these are the right submodules

$$\Gamma^k_S = \bigoplus_{\gamma \in \Gamma^k} \gamma e_{\tau(\gamma)} S e_{\tau(\gamma)} \quad \text{and} \quad {}_i\Gamma^k_S = \bigoplus_{\gamma \in {}_i\Gamma^k} \gamma e_{\tau(\gamma)} S e_{\tau(\gamma)}.$$

This module structure will play an important role in creating projective resolutions.

Definition IV.1. *For a subset $X \subseteq P$, let $X \circ \mathcal{M}$ be the set*

$$X \circ \mathcal{M} = \{(p,\ \alpha) \in X \times \mathcal{M} |\ \tau(p) = \sigma(\alpha)\}.$$

Each element of $X \circ \mathcal{M}$ is a path of P written with a specific factorization. If $X \subset {}_iP$ for some i, then we may use the suitable ordering $<$ on P to compare any two elements of $X \circ \mathcal{M}$. That is

$$(p_1,\ \alpha_1) \leq (p_2,\ \alpha_2) \Leftrightarrow p_1 \cdot \alpha_1 \leq p_2 \cdot \alpha_2.$$

We will utilize this comparison in the context of modules.

Note that since $\pi(\mathcal{M})S = R$, there is a natural S-S-bimodule structure on $\pi(\mathcal{M})S$. For any subset $X \subseteq P$ we may form the tensor product

$$X_S \otimes_S \pi(\mathcal{M})S = X_S \otimes_S R.$$

We note since $\pi(P_0) = E \subset S$,

$$p \otimes_S \pi(\alpha) \neq 0 \Leftrightarrow \tau(p) = \sigma(\alpha) \text{ and } (p,\ \alpha) \in X \circ \mathcal{M}.$$

Furthermore, since our binomial system $(\pi,\ T^*)$ is strong by Lemmas III.17 and III.20, $S\pi(\mathcal{M}) = R = \pi(\mathcal{M})S$ so we may write $r \in R$ as

$$r = \sum_{i=1}^{m} \sum_{j=1}^{\ell_i} t_{i,j} s_{i,j} = \sum_{i=1}^{m} \sum_{j=1}^{\ell_i} \pi(\alpha_{i,j}) s_{i,j}$$

where $t_{i,j} = \pi(\alpha_{i,j})$. We can then write elements of $X_S \otimes_S R$ as sums of $p \otimes_S r$, with $\sum_{i=1}^{m} \sum_{j=1}^{\ell_i} \pi(\alpha_{i,j}) s_{i,j} \in R$ and $(p,\ \alpha_{i,j}) \in X \circ \mathcal{M}$. We have the following.

Lemma IV.2. *If $X \subseteq {}_iP$ for some i, then $X_S \otimes_S R$ is a right R module with*

$$X_S \otimes_S R \cong \bigoplus_{p \in X} e_{\tau(p)} R.$$

Proof. Clearly, $X_S \otimes_S R$ is a right R-module under the natural right action of R. Since S is semisimple, the right S-module X_S is projective. Consider the map $\phi : e_{\tau(p)} S e_{\tau(p)} \to pS$ given by $s \to p \cdot s$. This is clearly an S-module isomorphism. Then for each $p \in X$, there is a right R-module isomorphism θ_p given by the composition:

$$pS \otimes_S R \to e_{\tau(p)} S e_{\tau(p)} \otimes_S R \cong e_{\tau(p)} S \otimes_S e_{\tau(p)} R \cong e_{\tau(p)} R.$$

The elements of ${}_iP$ are well-ordered under $<$. For each $p \in {}_iP$, write ω_p to be the ordinal corresponding to p under $<$.

Define a map

$$\psi : X_S \otimes_S R \to \bigoplus_{p \in X} e_{\tau(p)} R$$

by

$$\psi(p \otimes_S r) = (\ldots 0, \theta_p(p \otimes_S r), 0, \ldots)$$

where $\theta_p(p \otimes_S r)$ is an entry in the ω_p coordinate. Since each θ_p is an R-module isomorphism and $\theta_p(p \otimes_S R) = e_{\tau(p)} R$, it follow easily ψ is a surjective R-module homomorphism.

Suppose that $x = \sum_{p_j \in X} p_j \otimes_S r_j$ and

$$\psi(x) = \psi \left(\sum_{p \in X} p \otimes_S r \right) = (\theta_{p_1}(p_1 \otimes_S r_1), \theta_{p_2}(p_2 \otimes_S r_2), \ldots) = 0.$$

Then $\theta_{p_j}(p_j \otimes_S r_j) = 0$ for all j. But θ_{p_j} is an isomorphism for all j, so $p_j \otimes_S r_j = 0$ and $x = 0$. Hence, ψ is monic and ψ is an isomorphism. □

We now turn our attention to the specific case when $X = {}_i\Gamma^k$. Given $p \in {}_iP$, we may or may not be able to factor p as $p = \gamma \cdot \alpha$, with $\gamma \in {}_i\Gamma^k$ for some k, $\alpha \in \mathcal{M}$. By Lemma II.10, if such a factorization exists, it must be unique.

When p may be factored as an element of ${}_i\Gamma^k \circ \mathcal{M}$, we will use superscripts along with our previous convention to indicate this decompostion. Recall $p = \gamma_p^k \cdot \alpha_p^k$ indicates p has a (unique) decomposition $\gamma_p^k \cdot \alpha_p^k$ with $(\gamma_p^k,\ \alpha_p^k) \in \Gamma^k \circ \mathcal{M}$. We will use the set $\Gamma^k \circ \mathcal{M}$ to explore right R-modules.

Definition IV.3. For any $p \in P^*$, define the subset W_p^k of ${}_i\Gamma^k \otimes_S R$ to be

$$W_p^k = \text{r.span}_S\{\gamma \otimes_S \pi(\alpha) |\ (\gamma,\ \alpha) \in {}_i\Gamma^k \circ \mathcal{M} \text{ and } \gamma \cdot \alpha < p\}.$$

Lemma IV.4. *The subset W_p^k of ${}_i\Gamma^k \otimes_S R$ is a right S-module.* □

It will often be convenient to write an element $w \in {}_i\Gamma^k \otimes_S R$ relative to the subset W_p^k for a particular $p \in {}_iP$.

By $w \equiv \gamma_p^k \otimes \pi(\alpha_p^k)s \pmod{W_p^k}$ we mean

$$w = \gamma_p^k \otimes \pi(\alpha_p^k)s + \gamma_{q_1}^k \otimes \pi(\alpha_{q_1}^k)s_1 + \ldots + \gamma_{q_l}^k \otimes \pi(\alpha_{q_l}^k)s_l$$

with $(\gamma_{q_j}^k,\ \alpha_{q_j}^k) \in {}_i\Gamma^k \circ M,\ \gamma_p^k \cdot \alpha_p^k = p$ and $p > q_j = \gamma_j^k \cdot \alpha_j^k$ for $1 \leq j \leq l$.

The right S-modules W_p^k will figure prominently in the discussion of our main theorem. In the main theorem, we will define maps

$$\delta_k : {}_i\Gamma^k \otimes_S R \to {}_i\Gamma^{k-1} \otimes_S R$$

in order to construct a projective resolution for $\overline{e_iR}$. To do this, we must understand several properties of such maps.

IV.3. Delta Homomorphisms

Definition IV.5. By *cleft pair of homomorphisms*, $(\delta,\ \eta)$ we mean a family of homomorphisms

$$\delta_k : {}_i\Gamma^k \otimes_S R \to {}_i\Gamma^{k-1} \otimes_S R$$

$$\eta_k : \text{Ker } \delta_{k-1} \to {}_i\Gamma^k \otimes_S R$$

with δ_k a right R-module homomorphism and η_k a right S-module homomorphism, such that if $p = \gamma_p^k$ then

$$\delta_k(p \otimes \pi(v_{\tau(p)})) = \gamma_p^{k-1} \otimes \pi(\alpha_p^{k-1}) - \eta_{k-1}\delta_{k-1}(\gamma_p^{k-1} \otimes \pi(\alpha_p^{k-1})) \tag{IV.1}$$

$$\delta_k(p \otimes \pi(v_{\tau(p))})) \equiv \gamma_p^{k-1} \otimes \pi(\alpha_p^{k-1}) \pmod{W_p^{k-1}} \tag{IV.2}$$

$$\delta_k\eta_k(w) = w \tag{IV.3}$$

$$\text{For } w \in \text{Ker } \delta_k \text{ if } w \equiv \gamma_p^{k-1} \otimes \pi(\alpha_p^{k-1})s \pmod{W_p^{k-1}} \tag{IV.4}$$

$$\text{then } \eta_k(w) \equiv \gamma_p^{k} \otimes \pi(\alpha_p^{k})s \pmod{W_p^{k}}.$$

Lemma IV.6. *Let $(\delta,\ \eta)$ be a cleft pair of homomorphisms. Then Im $\delta_k \subseteq$ Ker δ_{k-1}.*

Proof. Consider a path $p = \gamma_p^k$ in ${}_i\Gamma^k$. Since δ_k is a right R-module homomorphism, $\delta_k(p \otimes r) = \delta_k(\gamma_p^k \otimes \pi(v_{\tau(p)})) \cdot r$. Then using the defining properties of δ_k (in particular

$\delta_k\eta_k(w) = w)$ we have

$$\begin{aligned}
\delta_{k-1}(\delta_k(p \otimes r)) &= \delta_{k-1}(\delta_k(\gamma_p^k \otimes \pi(v_{\tau(p)}))) \cdot r \\
&= [\delta_{k-1}(\gamma_p^{k-1} \otimes \pi(\alpha_p^{k-1}) - \eta_{k-1}\delta_{k-1}(\gamma_p^{k-1} \otimes \pi(\alpha_p^{k-1}))] \cdot r \\
&= [\delta_{k-1}(\gamma_p^{k-1} \otimes \pi(\alpha_p^{k-1})) - \delta_{k-1}(\eta_{k-1}\delta_{k-1}(\gamma_p^{k-1} \otimes \pi(\alpha_p^{k-1})))] \cdot r \\
&= [\delta_{k-1}(\gamma_p^{k-1} \otimes \pi(\alpha_p^{k-1})) - \delta_{k-1}(\gamma_p^{k-1} \otimes \pi(\alpha_p^{k-1}))] \cdot r \\
&= 0
\end{aligned}$$

The result now follows from the linearity of δ_k and that every element $x \in {}_i\Gamma^k \otimes_S R$ can be written as

$$x = \sum_{p \in {}_i\Gamma^m} p \otimes \pi(v_{\tau(\gamma)})r.$$

□

Lemma IV.7. *Let $\{\delta_k\}$ be the family of homomorphisms satisfying the properties of Definition IV.5. If $w \in W_p^k$ then $\delta_k(w) \in W_p^{k-1}$ for $k \geq 1$.*

Proof. Let $w \in W_p^k$. Then $w = \sum_{j=1}^{n} \gamma_{q_j}^k \otimes \pi(\alpha_{q_j}^k)s_{q_j}$ for some $(\gamma_{q_j}^k,\ \alpha_q^k) \in {}_i\Gamma^k \circ M$, $s_{q_j} \in S$, and $\gamma_{q_j}^k \cdot \alpha_{q_j}^k = q_j < p$, $1 \leq j \leq n$. Then

$$\delta_k(w) = \sum_{j=1}^{n} \delta_k(\gamma_{q_j}^k \otimes \pi(\alpha_{q_j}^k)s_{q_j}) = \sum_{j=1}^{n} \delta_k(\gamma_{q_j}^k \otimes e_{q_j})\pi(\alpha_{q_j}^k)s_{q_j}$$

where $e_{q_j} = \pi(v_{\tau(\gamma_{q_j}^k)})$.

For ease of notation, let $\hat{p}_j = \gamma_{q_j}^k$.

By definition of our family of maps, $\delta_k(\gamma^k_{q_j} \otimes e_{q_j}) \equiv \gamma^{k-1}_{\hat{p}_j} \otimes \pi(\alpha^{k-1}_{\hat{p}_j}) \pmod{W^{k-1}_{\hat{p}_j}}$.

Then

$$\begin{aligned}\delta_k(\gamma^k_{q_j} \otimes e_{q_j})\pi(\alpha^k_{q_j})s_{q_j} &\equiv \gamma^{k-1}_{\hat{p}_j} \otimes \pi(\alpha^{k-1}_{\hat{p}_j})\pi(\alpha^k_{q_j})s_{q_j} \pmod{W^{k-1}_{\hat{p}_j \cdot \alpha^k_{q_j}}} \\ &\equiv \gamma^{k-1}_{\hat{p}_j} \otimes \pi(\alpha^{k-1}_{\hat{p}_j} \cdot \alpha^k_{q_j})s_{q_j} \pmod{W^{k-1}_{\hat{p}_j \cdot \alpha^k_{q_j}}}.\end{aligned}$$

Note by Lemma II.10,

$$\gamma^{k-1}_{\hat{p}_j} \cdot \alpha^{k-1}_{\hat{p}_j} \cdot \alpha^k_{q_j} = \gamma^k_{q_j} \cdot \alpha^k_{q_j} = q_j = \gamma^{k-1}_{q_j} \cdot \alpha^{k-1}_{q_j}$$

for $1 \le j \le n$, implies $\gamma^{k-1}_{\hat{p}} = \gamma^{k-1}_{q_j}$ and $\alpha^{k-1}_{\hat{p}_j} \cdot \alpha^k_{q_j} = \alpha^{k-1}_{q_j}$.

Hence

$$\gamma^{k-1}_{\hat{p}_j} \otimes \pi(\alpha^{k-1}_{\hat{p}_j} \cdot \alpha^k_{q_j}) = \gamma^{k-1}_{q_j} \otimes \pi(\alpha^{k-1}_{q_j}).$$

So

$$\begin{aligned}\delta_k(\gamma^k_{q_j} \otimes e_{q_j})\pi(\alpha^k_{q_j})s_{q_j} &\equiv \gamma^{k-1}_{q_j} \otimes \pi(\alpha^{k-1}_{q_j})s_{q_j} \pmod{W^{k-1}_{\gamma^k_{q_j} \cdot \alpha^k_{q_j}}} \\ &\equiv \gamma^{k-1}_{q_j} \otimes \pi(\alpha^{k-1}_{q_j})s_{q_j} \pmod{W^{k-1}_{q_j}}.\end{aligned}$$

Since $q_j < p$, for $1 \le j \le n$.

$$\delta_k(\gamma^k_{q_j} \otimes e_{q_j})\pi(\alpha^k_{q_j})s_{q_j} \equiv \gamma^{k-1}_{q_j} \otimes \pi(\alpha^{k-1}_{q_j})s_{q_j} \pmod{W^{k-1}_{q_j}} \equiv 0 \pmod{W^{k-1}_p}$$

which implies

$$\delta_k(w) \equiv \sum_{j=1}^{n} \gamma^{k-1}_{q_j} \otimes \pi(\alpha^{k-1}_{q_j})s_{q_j} \pmod{W^{k-1}_p} \equiv 0 \pmod{W^{k-1}_p}$$

and $\delta_k(w) \in W^{k-1}_p$. □

IV.4. Main Theorem

Theorem IV.8. *Let R be a basic cleft binomial ring with strong binomial system $(\pi,\ T^*)$. Assume we have chosen a suitable ordering $<$ on P^* that respects the binomial system with $\Gamma^2 \subset P$ defined as before. For each i, $1 \le i \le n$ and let ${}_iR_k = {}_i\Gamma^k_S \otimes_S R$. Then there is a cleft pair of homomorphisms $(\delta,\ \eta)$ such that*

$$\begin{tikzcd}
0 & \overline{e_iR} \arrow[l] \arrow[r, bend right, "\eta_0"'] & {}_iR_0 \arrow[l, "\delta_0"'] \arrow[r, bend right, "\eta_1"'] & {}_iR_1 \arrow[l, "\delta_1"'] \arrow[r, bend right, "\eta_2"'] & {}_iR_2 \arrow[l, "\delta_2"'] \arrow[r, bend right] & \cdots \arrow[l]
\end{tikzcd}$$

is an exact sequence of right R-modules.

Proof. As in the finite dimensional algebra case (Theorem 2.7 of [4]), we will proceed by induction. We shall construct a pair of homomorphisms $(\delta,\ \eta)$ satisfying Definition IV.5.

Define $\delta_0 : {}_iR_0 \to \overline{e_iR}$ by

$$\delta_0(v_i \otimes r) = r + J \in \overline{e_iR}.$$

Noting that $e_ir + J \neq J$ implies $r = e_ire_i \in S$ we define $\eta_0 : \overline{e_iR} \to {}_iR_0$ by

$$\eta_0(e_ir + J) = \eta_0(e_ire_i + J) = v_i \otimes e_ire_i.$$

One easily checks that δ_0, η_0 satisfy Equations IV.3 and IV.4 of Definition IV.5.

For $p = \gamma_p^1 \in {}_i\Gamma^1$, define $\delta_1 : R_1 \longrightarrow R_0$ by

$$\delta_1(\gamma_p^1 \otimes r) = v_i \otimes \pi(\gamma_p^1)r.$$

Extend by linearity to define δ_1 on all of R_1. An easy check shows this is a well-defined right R-module homomorphism.

Since $p = \gamma_p^1 = v_i \cdot \alpha_p^0$

$$\delta_1(p \otimes e_{\tau(p)}) = v_i \otimes \pi(\alpha_p^0) = v_i \otimes \pi(\alpha_p^0) - \eta_0\delta_0(v_i \otimes \pi(\alpha_p^0))$$

the map δ_1 satisfies Equation IV.1 of Definition IV.5. Note that

ker $\delta_0 = \{v_i \otimes r | \ r \in J\}$. For $r \in J$, we have

$$r = \sum_{p \in {}_iM \setminus \{v_i\}} \pi(p)s_p, \ \ s_p \in S$$

so the map $\delta_1 : R_1 \to R_0$ is onto.

Define $\eta_1 :$ ker $\delta_0 \to R_1$ by

$$\eta_1(v_i \otimes \pi(p)s) = \gamma_p^1 \otimes \pi(\alpha_p^1)s.$$

Extend η_1 to all of ker δ_0 by linearity. For $p \in \mathcal{M}$ we have $p = v_i \cdot p = v_i \cdot \alpha_p^0$, so $p = \alpha_p^0$. If $p \neq v_i$, then $p = \gamma_p^1 \cdot \alpha_p^1$ for some $\gamma_p^1 \in P^* \setminus \{v_i\}$. This gives:

$$\begin{aligned}\delta_1\eta_1(v_i \otimes \pi(p)s) &= \delta_1(\gamma_p^1 \otimes \pi(\alpha_p^1)s) = v_i \otimes \pi(\gamma_p^1)\pi(\alpha_p^1)s \\ &= v_i \otimes \pi(\gamma_p^1 \cdot \alpha_p^1)s = v_i \otimes \pi(p)s.\end{aligned}$$

So Equation IV.3 of Definition IV.5 is valid for $k = 1$.

For $p = \gamma_p^1 = \alpha_p^1 \in {}_i\Gamma^1$, $p = v_i \cdot \alpha_p^0 = \gamma_p^1 \cdot v_{\tau(p)}$ then $v_i = \gamma_p^0$ and $\alpha_p^0 = v_{\tau(p)}$. So $\delta_1(p \otimes e_{\tau(p)}) = v_i \otimes \pi(\alpha_p^0) \equiv \gamma_p^0 \otimes \pi(\alpha_p^0) \pmod{W_0^\alpha}$ and

$$\eta_1(v_i \otimes \pi(p) \cdot s) = \gamma_p^1 \otimes e_{\tau(p)} \cdot s \equiv \gamma_p^1 \otimes \pi(\alpha_p^1) \cdot s \pmod{W_p^1}.$$

Hence Equations IV.2 and IV.4 of Definition IV.5 are valid for $k = 1$.

The above work establishes a base case for our induction. In assuming the induction hypothesis, for $0 \leq j \leq k$, we have defined a family of maps

$$\delta_j : R_j \to R_{j-1}$$

$$\eta_j : \ker\, \delta_{j-1} \to R_j$$

with δ_j a right R-module homomorphism and η_j a right S-module homomorphism, satisfying Definition IV.5.

Define $\delta_{k+1} : R_{k+1} \to R_k$ by

$$\delta_{k+1}(p \otimes r) = [\gamma_p^k \otimes \pi(\alpha_p^k) - \eta_k \delta_k(\gamma_p^k \otimes \pi(\alpha_p^k))]r$$

where $p = \gamma_p^k \cdot \alpha_p^k \in {}_i\Gamma^{k+1}$ with $\gamma_p^k \in {}_i\Gamma^k,\ \alpha_p^k \in \mathcal{M}$. One checks this is a well-defined R-module homomorphism satisfying Equation IV.1 of Definition IV.5. Using our induction hypothesis, we find that δ_{k+1} satisfies Equation IV.2 of Definition IV.5.

What remains is to define a map $\eta_{k+1} : \ker\, \delta_k \to R_{k+1}$ satisfying equations IV.3 and IV.4 of Definition IV.5.

For each $p \in {}_iP$, we will define a map

$$\eta_{k+1}^p : \ker\, \delta_k \cap W_p^k \to R_{k+1}.$$

Since $<$ gives rise to a well-ordering of ${}_iP$, we may define η_{k+1}^p successively in accordance to the order. This order applies to any subset of ${}_iP$. Consider the total order on ${}_i\Gamma^k$ and suppose $p \in {}_i\Gamma^k$ with successor $\hat{p}$.

In the process of defining each η_{k+1}^p, we will show that each successive map actually extends the previous. Eventually, this extension process will enable us to define η_{k+1}.

Proceed by transfinite induction. For $p = v_i$ we see that $W_{v_i}^k = 0$, so we have the zero map

$$\eta_{k+1}^{v_i} : \ker\, \delta_k \cap W_{v_i}^k \to R_{k+1}.$$

Then $\eta_{k+1}^{v_i}$ satisfies equations $\delta_{k+1}\eta_{k+1}^{v_i}(0) = 0$ and $\eta_{k+1}^{v_i}(0) = 0 \pmod{W_{v_i}^k}$. We have established a base case for defining maps η_{k+1}^p. Note that for $p \leq \hat{p}$, we have $W_{k+1}^p \subseteq W_{k+1}^{\hat{p}}$ for all k. Assume the induction hypothesis, we have defined maps

$$\eta_{k+1}^p : \ker\, \delta_k \cap W_p^k \to R_{k+1}$$

satisfying Equations IV.3 and IV.1 of Definition IV.5 such that η_{k+1}^p extends all preceding maps.

We must now define a map $\eta_{k+1}^{\hat{p}}$ satisfying Equations IV.3 and IV.4 of Definition IV.5 so that $\eta_{k+1}^{\hat{p}}$ is an extension of η_{k+1}^p. The right S-module map $\eta_{k+1}^{\hat{p}}$ will be defined on $\ker\, \delta_k \cap W_{\hat{p}}^k$. Let $w \in \ker\, \delta_k \cap W_{\hat{p}}^k$. Invoking the total order on ${}_i\Gamma^k$, we must have

$$w \equiv \gamma_p^k \otimes \pi(\alpha_p^k)s \pmod{W_p^k}.$$

By lemma IV.7 $\delta_k(W_p^k) \subseteq W_p^{k-1}$, so we have

$$0 = \delta_k(w) \equiv \delta_k([\gamma_p^k \otimes \pi(\alpha_p^k)] \cdot s) \pmod{W_p^{k-1}}$$

where $p = \gamma_p^k \cdot \alpha_p^k$ with $\gamma_p^k \in {}_i\Gamma^k,\ \alpha_p^k \in \mathcal{M}$.

Set $q = \gamma_p^k \in {}_i\Gamma^k$. By definition of Γ^k we have a factorization $q = \gamma_q^{k-1} \cdot \alpha_q^{k-1}$ with $\gamma_q^{k-1} \in \Gamma^{k-1}$.

By equation IV.2 of Definition IV.5,

$$\delta_k(\gamma_p^k \otimes \pi(\alpha_p^k) \cdot s) \equiv \gamma_q^{k-1} \otimes \pi(\alpha_q^{k-1}) \cdot \pi(\alpha_p^k)s \quad (\text{mod } W_p^{k-1}).$$

Combining the above equations, it follows

$$0 = \delta_k(w) \equiv \gamma_q^{k-1} \otimes \pi(\alpha_q^{k-1} \cdot \alpha_p^k)s \quad (\text{mod } W_p^{k-1}).$$

If $s = 0$, then $w \in W_p^k$ and by our induction hypothesis, we have $\eta_k^{\hat{p}}(w) = \eta_k^p(w)$ defined. Assume $s \neq 0$. If $\alpha_q^{k-1} \cdot \alpha_p^k \in \mathcal{M}$, then $\gamma_q^{k-1} \otimes \pi(\alpha_q^{k-1} \cdot \alpha_p^k)s \equiv 0 \ (\text{mod } W_p^{k-1})$ implies that $\gamma_q^{k-1} \cdot \alpha_q^{k-1} \cdot \alpha_p^k < p$. But $p = \gamma_q^{k-1} \cdot \alpha_q^{k-1} \cdot \alpha_p^k$, so we must have $\alpha_q^{k-1} \cdot \alpha_p^k \in P \setminus \mathcal{M}$.

We have just given a decomposition of p as $p = \gamma_q^{k-1} \cdot \alpha_q^{k-1} \cdot \alpha_p^k$ with $\gamma_q^{k-1} \cdot \alpha_q^{k-1} = q \in {}_i\Gamma^k$, $\gamma_q^{k-1} \in \Gamma^{k-1}$ and $\alpha_q^{k-1} \cdot \alpha_p^k \in P \setminus \mathcal{M}$.

Now invoking Lemma II.10, we may write $p = \gamma_{\hat{p}}^{k+1} \cdot \alpha_p^{k+1}$ where $\gamma_p^{k+1} \in {}_i\Gamma^{k+1}$, $\alpha_p^{k+1} \in P^*$. We will use this factorization of p to define $\eta_{k+1}^{\hat{p}}$. Note that γ_p^k is a proper subpath of p implying α_p^{k+1} is a proper subpath of α_p^k. But $\alpha_p^k \in \mathcal{M}$, so $\alpha_p^{k+1} \in \mathcal{M}$ as well, and our use of notation is consistent. This says that $p = \gamma_p^{k+1} \cdot \alpha_p^{k+1}$

with $\gamma_p^{k+1} \in {}_i\Gamma^{k+1}$ and $\alpha_p^{k+1} \in \mathcal{M}$. Consider the element $y = w - \delta_{k+1}(\gamma_p^{k+1} \otimes \alpha_p^{k+1})s$. Then $\delta_k(y) = \delta_k(w) = 0$ since $w \in \ker\, \delta_k \cap W^k_{\hat{p}}$. Also $w - \delta_{k+1}(\gamma_p^{k+1} \otimes \alpha_p^{k+1})s \equiv 0 \pmod{W_p^k}$. So $y \in \ker\, \delta_k \cap W_p^k$.

Define $\eta^{\hat{p}}_{k+1} : \ker\, \delta_k \cap W_k^{\hat{p}} \to R_{k+1}$ by

$$\eta^{\hat{p}}_{k+1}(w) = \gamma_p^{k+1} \otimes \pi(\alpha_p^{k+1})s + \eta^p_{k+1}(y).$$

This is well-defined since $y \in \ker\, \delta_k \cap W_k^p$ and by our induction hypothesis, the map η^p_{k+1} exists.

An easy check shows that $\eta^{\hat{p}}_{k+1}$ extends η^p_{k+1} and that Equations IV.3 and IV.4 of Definition IV.5 hold for $\eta^{\hat{p}}_{k+1}$.

When $\hat{p}$ corresponds to a limit ordinal in $(P,\ <)$ we have

$$W^k_{\hat{p}} = \bigcup_{p<\hat{p}} W^k_p$$

In this case, we define $\eta^{\hat{p}}_{k+1}$ to be $\bigcup\limits_{p<\hat{p}} \eta^p_{k+1}$. Again, we have $\eta^{\hat{p}}_{k+1}$ extending η^p_{k+1} and Equations IV.3 and IV.4 of Definition IV.5 hold for $\eta^{\hat{p}}_{k+1}$. Since $\delta_k(\eta_k(w)) = w$ for all $w \in \ker\, \delta_k$ we have δ_k is surjective for all k and the resolution is exact. Therefore, $(\delta,\ \eta)$ is a cleft pair for the resolution

$$\begin{tikzcd}
0 & \overline{e_iR} \arrow[l] \arrow[r, bend right, "\eta_0"'] & {}_iR_0 \arrow[l, "\delta_0"'] \arrow[r, bend right, "\eta_1"'] & {}_iR_1 \arrow[l, "\delta_1"'] \arrow[r, bend right, "\eta_2"'] & {}_iR_2 \arrow[l, "\delta_2"'] \arrow[r, bend right] & \cdots \arrow[l] .
\end{tikzcd}$$

□

Corollary IV.9. *There is a projective resolution for R/J as a right R-module given by*

$$\begin{tikzcd}
0 & R/J \arrow[l] \arrow[r, bend right, "\eta_0"'] & \Gamma_S^0 \otimes_S R \arrow[l, "\Delta_0"'] \arrow[r, bend right, "\eta_1"'] & \Gamma_S^1 \otimes_S R \arrow[l, "\Delta_1"'] \arrow[r, bend right, "\eta_2"'] & \Gamma_S^2 \otimes_S R \arrow[l, "\Delta_2"'] \arrow[r, bend right] & \cdots . \arrow[l]
\end{tikzcd}$$

Proof. Take the direct sum of the resolutions from Theorem IV.8 over all i. $\square$

CHAPTER V

APPLICATIONS AND EXAMPLES

We now explore some consequences of Theorem IV.8.

V.1. Monomial Algebras

Definition V.1. Let A be a split finite dimensional K-algebra A with right quiver Γ and path semigroup P. We say A is a *monomial algebra* if $A \cong KP/I$ where I is an ideal generated by relations of the form λp, $\lambda \in K$ and p a path in P of length ≥ 2.

A monomial algebra is cleft with subring $S = \text{span}_K\{\Gamma^0 + I\}$. By Theorem 3.2 of [11], a monomial algebra (ring) is a binomial ring with strong binomial system. As was done earlier in [4], we may apply Theorem IV.8 to monomial algebras.

One consequence arising from Theorem 2.3 of [5] is that homological dimension of monomial algebras does not depend on the choice of field. Moreover, it is relatively easy to compute the global dimension of a monomial algebra, and algorithms exist that do this (see, for example, [4], [8], and [13]). Next we compare the homological dimension of cleft binomial rings with that of monomial algebras. We begin with the following.

Let R be a cleft binomial ring with strong binomial system $(\pi_1,\ T^*)$ consistent with a suitable order $<_1$ on the free path semigroup P. Let $\Gamma^2_{<_1}$ be the obstruction set arising from $<_1$ and denote by KP the path algebra over an arbitrary field K. If $A = KP/I$ where I is the two-sided ideal generated by $\Gamma^2_{<_1}$ in KP, then A is a monomial algebra with associated path semigroup P. There is a map

$$\pi_2 : P \longrightarrow A$$

arising from the surjection $KP \longrightarrow A$. Given such a map, any ordering $<_2$ on P gives rise to a new obstruction set $\Gamma^2_{<_2}$. However, we have the following which is given in [4] without proof.

Lemma V.2. *If R, A, and P are as above, then the two obstruction sets are equal, i.e., $\Gamma^2_{<_1} = \Gamma^2_{<_2}$.*

Proof. Let $\mathcal{M}_1,\ \mathcal{M}_2$ be subsets of P arising from $(\pi_1,\ <_1)$ and $(\pi_2,\ <_2)$, respectively. Since A is monomial, paths p and q are equivalent under π_2 if and only if

$$\pi_2(p) = \pi_2(q) = 0 + I.$$

This implies p and q must have subpaths belonging to $\Gamma^2_{<_1}$. So

$$\mathcal{M}_2 = \{p \in P |\text{ no subpath of } p \text{ belongs to } \Gamma^2_{<_1}\}.$$

Since $\mathcal{M}_1$ is closed under subpaths, we have $\mathcal{M}_1 \subseteq \mathcal{M}_2$. Since $p \in \Gamma^2_{<_1}$ has all proper subpaths belonging to $\mathcal{M}_1$, by definition of $\Gamma^2_{<_2}$ we have $p \in \Gamma^2_{<_2}$ and $\Gamma^2_{<_1} \subseteq \Gamma^2_{<_2}$. Let $p \in \Gamma^2_{<_2}$. So $\pi_2(p) = 0 + I$ implying p has a subpath q with $q \in \Gamma^2_{<_1}$. But then

$q \in \Gamma^2_{<_2}$ and $p = q \cdot \beta$ implies $p = q$ and $\beta = v_{\tau(p)}$ by definition of $\Gamma^2_{<_2}$. Hence $\Gamma^2_{<_1} = \Gamma^2_{<_2}$. □

So the sets Γ^k for $A = KP/I$ are invariant under different suitable orderings. This makes calculations with monomial algebras attractive. We now would like to investigate circumstances under which the resolution of Theorem IV.8 is minimal. We begin with the following:

Lemma V.3. *In the resolution of Theorem IV.8, ker* δ_k *is superfluous in* $\Gamma^k_S \otimes_S R$ *for* $k = 0, 1$.

Proof. We begin by examining the maps δ_k, $k = 0, 1$. Note that $(\Gamma^k_S \otimes_S R)J = \Gamma^m_S \otimes_S J$. Recall ${}_i\Gamma^0 = \{v_i\}$ and consider $0 \neq v_i \otimes_S r \in {}_i\Gamma^0_S \otimes_S R$ (in particular, $r = e_i r$) with

$$\delta_0(v_i \otimes r) = \pi(v_i)r + J = e_i r + J = 0.$$

If $r \in S$, then $e_i r \notin J$, hence $e_i r + J \neq 0$. So $r \in J$ and ker $\delta_0 = \Gamma^0_S \otimes_S J$.

Now consider ker δ_1. Note that ${}_i\Gamma^1$ is the set of arrows of P with initial vertex v_i. Suppose $a_j \in P_1$, $\alpha_j \in \mathcal{M}$, $s_j \in S$ for all j so that

$$\begin{aligned}\delta_1\left(\sum_{j=1}^{n} a_j \otimes_s \pi(\alpha_j)s_j\right) &= \sum_{j=1}^{n} v_i \otimes_S \pi(a_j)\pi(\alpha_j)s_j \\ &= v_i \otimes_S \left(\sum_{j=1}^{n} \pi(a_j)\pi(\alpha_j)s_j\right) = 0.\end{aligned}$$

So $\sum_{j=1}^{n} \pi(a_j)\pi(\alpha_j)s_j = 0$. If $\alpha_j \neq v_{\tau(p_j)}$, for all j, then $\pi(\alpha_j) \in J$ and $\ker \delta_1 \leq {}_i\Gamma_S^1 \otimes_S J$ is superfluous. We may assume that $\alpha_n = v_{\tau(p_n)}$ and

$$\pi(a_n)(-s_n) = \sum_{j=1}^{n-1} \pi(a_j)\pi(\alpha_j)s_j.$$

Then

$$\pi(a_n)(-s_n) + J^2 = \sum_{j=1}^{n-1} \pi(a_j)\pi(\alpha_j)s_j + J^2.$$

If $\alpha_j \neq v_{\tau(p_j)}$ then $\pi(a_j)\pi(\alpha_j) \in J^2$ and $\pi(a_j)\pi(\alpha_j)s_j + J^2 = 0 + J^2$. But $\alpha_j = v_{\tau(p_j)}$ implies $a_n(-s_n) + J^2$ generates the same submodule of J/J^2 as $\sum_{j=1}^{n-1} \pi(a_j)s_j + J^2$. This is impossible unless $a_j = a_n$ for all j, in which case $\sum_{j=1}^{n} s_j + J^2 = 0 + J^2$ implying $s_j = 0$ whenever $\alpha_j \neq v_{\tau(a_j)} \Leftrightarrow \pi(\alpha_j) \in J$. Hence $\ker \delta_1 \leq \Gamma_S^1 \otimes_S J$ is superfluous.

□

We now show in certain situations the resolution of Theorem IV.8 continues in this vein. The following was given in [4] with an alternate proof.

Lemma V.4. *For a monomial algebra $A = K\Gamma/I$ the resolution of Theorem IV.8 is minimal.*

Proof. We begin by showing $\delta_k(p \otimes e_{\tau(p)}) = \gamma_p^k \otimes \pi(\alpha)$ for all $p \in \Gamma^k$. Lemma V.3 establishes this result for $k = 1$. Now assume the result holds up to δ_k and consider δ_{k+1}. By our inductive definition of δ_{k+1},

$$\delta_{k+1}(p \otimes e_{\tau(p)}) = \gamma_p^k \otimes \pi(\alpha_p^k) - \eta_k(\delta_k(\gamma_p^k \otimes \pi(\alpha_p^k)))$$

Let $q = \gamma_p^k$. By definition of δ_k and our induction hypothesis,

$$\begin{aligned}\delta_k(\gamma_p^k \otimes \pi(\alpha_p^k)) &= \delta_k(q \otimes \pi(\alpha_p^k)) \\ &= \gamma_q^{k-1} \otimes \pi(\alpha_q^{k-1})\pi(\alpha_p^k) \\ &= \gamma_q^{k-1} \otimes \pi(\alpha_q^{k-1} \cdot \alpha_p^k)s\end{aligned}$$

for some $s \in S$. By definition of $p \in \Gamma^{k+1}$, we have $p = q \cdot \alpha_p^k = \gamma_q^{k-1} \cdot \alpha_q^{k-1} \cdot \alpha_p^k$ with $\alpha_q^{k-1} \cdot \alpha_p^k \notin \mathcal{M}$. Since A is monomial, the map

$$\pi : P \longrightarrow A$$

has the property that for all $p \notin \mathcal{M}$, $\pi(p) = 0$. Hence

$$\delta_k(\gamma_p^k \otimes \pi(\alpha_p^k)) = \gamma_q^{k-1} \otimes \pi(\alpha_q^{k-1} \cdot \alpha_p^k)s = 0$$

So by induction we have

$$\delta_{k+1}(p \otimes e_{\tau(p)}) = \gamma_p^k \otimes \pi(\alpha_p^k) - \eta_k(0) = \gamma_p^k \otimes \pi(\alpha_p^k).$$

Now suppose $\delta_{k+1}\left(\sum_{j=1}^{n} p_j \otimes r_j\right) = 0$ with $r_j = \pi(\alpha_j)s_j \in R$ for all j. From the above result

$$\begin{aligned}\delta_{k+1}\left(\sum_{j=1}^{n} p_j \otimes r_j\right) &= \sum_{j=1}^{n} \gamma_{p_j}^k \otimes \pi(\alpha_{p_j}^k)r_j \\ &= \sum_{j=1}^{n} \gamma_{p_j}^k \otimes \pi(\alpha_{p_j}^k)\pi(\alpha_j)s_j = 0.\end{aligned}$$

Without loss of generality, we may assume $\gamma_{p_j}^k = \gamma_p^k$ for all j and $\pi(\alpha_{p_j}^k)r_j \neq 0$. Since A is monomial

$$\pi(\alpha_{p_j}^k)r_j = \pi(\alpha_{p_j}^k) \cdot \pi(\alpha_j)s_j \neq 0 \Rightarrow \alpha_{p_j}^k \cdot \alpha_j \in \mathcal{M}.$$

Assume $r_n \in S$. Then

$$\gamma_p^k \otimes \pi(\alpha_{p_n})(-s_n) = \sum_{j=1}^{n-1} \gamma_p^k \otimes \pi(\alpha_{p_j}^k \cdot \alpha_j)\hat{s}_j$$

for some $\hat{s}_j \in S$. This implies $\pi(\alpha_{p_n})(-s_n) = \sum\limits_{j=1}^{n-1} \pi(\alpha_{p_j}^k \cdot \alpha_j)\hat{s}_j$. Since $\pi(\mathcal{M})$ forms a basis for A over K and $\hat{s}_j \in eSe = K$ for some idempotent e, this is possible only if $\alpha_{p_n} = \alpha_{p_j}^k \cdot \alpha_j$. Then $p_j \cdot \alpha = \gamma_p^k \cdot \alpha_{p_j}^k \cdot \alpha_j = p_n$ with $p_j,\ p_n \in \Gamma^2$ for all j. By the definition of Γ^2, this implies $\alpha_j \in \Gamma^0$ and $p_n = p_j$ for all j. Hence $\sum\limits_{j=1}^{n} p_j \otimes r_j = \sum\limits_{j=1}^{n} p_j \otimes s_j = 0 \Rightarrow s_j = 0$ for all j. So the only way such a sum is in the kernel of δ_{k+1} is if $r_j \in J(A)$ for all j. Thus ker δ_{k+1} is superfluous for all k. □

The above lemma leads to a comparison between the global dimension of monomial algebras and that of cleft binomial rings with strong binomial systems.

Corollary V.5. *Let R be a basic cleft binomial ring with left quiver Γ, strong binomial system $(\pi,\ T^*)$ and compatible suitable ordering $<$ on the associated path semigroup P. Let K be any field and suppose that Γ^2 is a subset of P obtained from $<$ as before. Let I be the ideal of $K\Gamma$ generated by Γ^2 and denote by A the algebra $A = K\Gamma/I$. Then A is a monomial algebra and*

$$gl.dim(R) \leq gl.dim(A).$$

Proof. As I is an ideal consisting of paths of length ≥ 2, in the path semigroup of Γ, it follows A is a monomial algebra. Let $E = \{e_1, \ldots, e_n\}$ be a basic set of idempotents for A and set $S = \text{span}_K\{E\}$. Let ${}_iA_j = K_j\Gamma_S \otimes_S A$. By Lemma V.4, the resolution

$$\begin{tikzcd}
0 & \overline{e_iA} \arrow[l] \arrow[r, bend right, "\eta_0"'] & {}_iA_0 \arrow[l, "\delta_0"'] \arrow[r, bend right, "\eta_1"'] & {}_iA_1 \arrow[l, "\delta_1"'] \arrow[r, bend right, "\eta_2"'] & {}_iA_2 \arrow[l, "\delta_2"'] \arrow[r, bend right] & \cdots \arrow[l]
\end{tikzcd}$$

is minimal for $i = 1, \ldots, n$.

The algebra A has global dimension bounded above by B implies $\mathrm{pdim}(\overline{e_iA}) \leq B$ for $i = 1, \ldots, n$. Then it follows that $\Gamma^i_{B+1} = \emptyset$ for $i = 1, \ldots, n$.

Since the resolutions of Theorem IV.8 for $\overline{e_iR}$ are constructed using ${}_i\Gamma^k$,

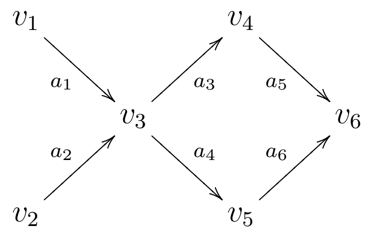
$\mathrm{pdim}(\overline{e_iR}) \leq B$ for $i = 1, \ldots, n$.

This implies $\mathrm{gl.dim}(R) \leq B$. $\square$

V.2. Examples

In [4], Anick and Green compute bounds on global dimension of several examples using their main theorem. In all of the following examples, we diverge from our established convention a write the product in the associated semigroups using juxtaposition. Two examples (Example 3.6 and 3.7) are binomial algebras (rings), so our Theorem IV.8 applies. In particular, these examples show the limits to the utility of Theorem IV.8. We summarize their results.

Example V.6. Let Γ be the following directed graph.



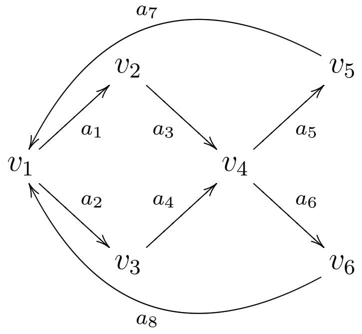
Let I be the ideal of $K\Gamma$ generated by the relations $\{a_1a_3,\ a_2a_4,\ a_3a_5 - a_4a_6\}$. Then $A = K\Gamma/I$ is a binomial algebra (cleft binomial ring with strong binomial system). An easy check shows that $\text{gldim}(A) = 2$. However, if we impose any suitable ordering where $a_3a_5 < a_4a_6$ we have

$$\Gamma^2 = \{a_1a_3,\ a_2a_4,\ a_4a_6\} \quad \text{and} \quad \Gamma^3 = \{a_2a_4a_6\}.$$

Hence, the monomial algebra $K\Gamma/\langle\Gamma^2\rangle$ has dimension three and we see the resolution of Theorem IV.8 does not give the precise dimension of A.

It is also possible for the global dimension of a binomial algebra to be finite, while the bound determined by Theorem IV.8 is infinite.

Example V.7. Let Γ be the following directed graph.



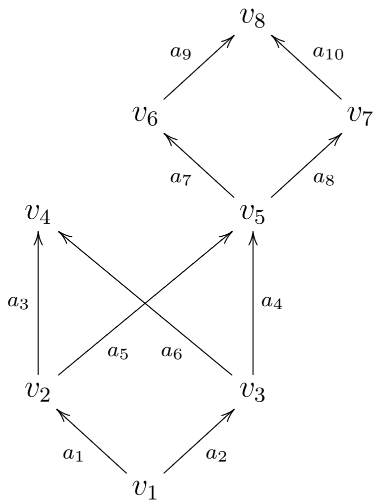
Let I be the ideal of $K\Gamma$ generated by the relations

$$\{a_3a_5a_7a_1,\ a_3a_6a_8a_2,\ a_4a_5a_7a_1,\ a_4a_6a_8a_2,\ a_1a_3 - a_2a_4\}$$

and consider $A = K\Gamma/I$. As shown by Anick and Green, gldim$(A) = 3$. However, any suitable ordering leads to $\Gamma^k \neq \emptyset$ for all k. Hence the monomial algebra determined by any Γ^2 has infinite global dimension.

We now consider examples that need not be algebras. D'Ambrosia (see Example 3.6 of [6]) has shown there are square-free rings that are not algebras. We utilize this work in the following example

Example V.8. Let Γ be the following directed graph.



Let P_0 and P_1 denote the vertices and arrows of Γ. Associated to Γ is the free path semigroup P generated by $(P_0,\ P_1)$. For this example, we suppress our previous use of $\cdot$ and write the operation of P using juxtaposition. Impose a suitable ordering $<$ on P. Adapting Corollary 3.5 of [6], there is a square-free ring R with quiver Γ such that R is not an algebra. We shall consider such an R with binomial system $(\pi,\ T^*)$

consistent with $<$ such that

$$\mathcal{M} = \pi^{-1}(T^*) = P_0 \cup P_1 \cup \{a_1a_3,\ a_1a_5,\ a_1a_5a_7,\ a_1a_5a_8,\ a_7a_9, a_4a_7a_9,\ a_5a_7a_9,\ a_1a_5a_7a_9\}.$$

We have $\pi(a_1a_3)R = \pi(a_2a_6)R$, $\pi(a_1a_5)R = \pi(a_2a_4)R$ and

$$K_\pi = \{p \in P|\ \pi(p) = 0\} = \{a_8a_{10}\}.$$

It follows

$$\Gamma^2 = \{a_2a_6,\ a_2a_4,\ a_8a_{10}\} \qquad \text{and} \qquad \Gamma^k = \emptyset,\ k \geq 3.$$

Hence, the associated monomial algebra $A = K\Gamma/\langle\Gamma^2\rangle$ has global dimension two. An easy check shows that $\mathrm{gldim}(R) = 2$ as well.

CHAPTER VI

RINGS WITH LOCAL UNITS

We now investigate a class of rings whose members need not have an identity, but do have many properties discussed earlier. Examples of such rings have been explored in other contexts (see [1] and [2] for example). We begin with some basic definitions. A set $\mathcal{U} \subset R$ of nonzero idempotents is a *set of local units* for R if for each pair $x, y \in R$ there exists $u \in \mathcal{U}$ such that $x, y \in uRu$. If $\mathcal{U}$ is a set of local units for R, then each uRu with $u \in \mathcal{U}$ is a unital ring with unit u. It follows easily that the set $\{uRu : u \in \mathcal{U}\}$ is a directed system of unital subrings of R, directed by $\subseteq$, with $R = \cup_{u \in \mathcal{U}} uRu$.

Let R be a ring and E a set of pairwise orthogonal idempotents in R. The set $\mathcal{U} = \mathcal{U}_E$ of all orthogonal finite sums from E is an *idempotent lattice generated by* E. If the idempotents in E are primitive, and $\mathcal{U}$ is a set of local units for R, then R has an *atomic set* of $\mathcal{U}$ of local units with *atoms* E. In such cases, we will often say that E is a *set of atoms* for R. We say that R is *atomic* in case it has an atomic set of local units.

Let R be a ring with local units and atomic set E. Then R is *locally artinian* if for every element u in the idempotent lattics $\mathcal{U}_E$ generated by E the unital ring uRu is artinian. Let R be locally artinian with atomic set E. Then for each $e \in E$

the left and right modules Re and eR are indecomposable, projective, and

$$_RR = \bigoplus_{e \in E} Re \qquad \text{and} \qquad R_R = \bigoplus_{e \in E} eR.$$

We will assume any locally artinian ring R is basic, i.e., for all $e_i,\ e_j \in E$, $e_iR \cong e_jR$ implies $e_i = e_j$. Although the radical $J = J(R)$ of R need not be nilpotent, for each $u \in \mathcal{U}_E$ the radical uJu of the artinian ring uRu is nilpotent. A subset I of a ring R is *left T-nilpotent* if for every sequence $a_1,\ a_2, \ldots$ in I, there is an n such that

$$a_1 \ldots a_n = 0.$$

Similarly, I is *right T-nilpotent* if for each $a_1,\ a_2, \ldots$ in I, we have an n such that

$$a_n \ldots a_1 = 0.$$

Henceforth, we will assume that J is both left and right T-nilpotent. By Lemma 28.3. of [3] we have JM a superfluous submodule of M for every nonzero module M. Also, the radical has the property

$$\bigcap_{i=1}^{\infty} J^i = 0$$

which implies $J^i \neq J^{i+1}$, for all i.

Let R be a locally artinian ring with atomic set E, set of local units $\mathcal{U}_E = \mathcal{U}$ and radical J. The ring R/J is a *semisimple ring with local units*, i.e, for every $u \in \mathcal{U}$ the cyclic modules $\overline{Ru} = Ru/Ju$ and $\overline{uR} = uR/uJ$ are semisimple. For all i, J^i/J^{i+1} is a semisimple R-module and we can find a collection $T_i \subset J^i \setminus J^{i+1}$ such that

$$_RJ^i/J^{i+1} = \bigoplus_{t \in T_i} (Rt + J^{i+1})/J^{i+1}$$

is a semisimple decomposition of J^i/J^{i+1}. In particular, we may assume that $T_0 = E$. If for each $t \in T_i$, there are unique elements of our atomic set $e, f \in E$ such that $t = etf$, we have the following result, proved by Sklar for unital binomial rings.

Lemma VI.1. *Let R be a locally artinian ring with atomic set E and let $T_i \subset R$ such that*

$$_RJ^i/J^{i+1} = \bigoplus_{t \in T_i} (Rt + J^{i+1})/J^{i+1}.$$

If $t = etf$ for primitive idempotents $e,\ f \in E$, then as simple left R-modules

$$(Rt + J^{i+1})/J^{i+1} \cong Rt/Jt \cong Re/Je$$

A symmetric statement is also true for simple right R-modules.

Proof. Fix $t \in T_i$. Let $\rho : Re \longrightarrow Rt$ be the epimorphism given by right multiplication by t. The projective Re is local, hence Je is the unique maximal submodule and $Rt \neq 0$ implies Ker $\rho \subseteq Je$. Since Je is superfluous in Re, we have $(R,\ \rho)$ a projective cover for Rt. As in the unital case, this implies $Rt/Jt \cong Re/Je$. Next, we notice the R-linear map $\phi : Rt \longrightarrow (Rt + J^{i+1})/J^{i+1}$ is a nonzero epimorphism with Ker $\phi \subseteq Jt$. Since we also have $Jt \subset$ Ker ϕ we see $Rt/Jt \cong (Rt + J^{i+1})/J^{i+1}$. The symmetric statement about right R-modules follows easily as above. □

Consider R a locally artinian ring with atomic set E for which we have chosen sets $T_i \subset J^i \setminus J^{i+1}$. We may assume for each i and $t \in T_i$, $t = etf$ for some primitive idempotents $e,\ f \in E$ and consider T_0 and T_1. The property $t = etf$ for $t \in T_1$ determines two set maps $\sigma_R, \tau_R : T_1 \longrightarrow T_0 = E$ given by $\sigma_R(t) = e$ and $\tau_R(t) = f$.

Consider the free path semigroup P generated by $(T_0,\ T_1)$ with $\sigma_P = \sigma_R$ and $\tau_P = \tau_R$. We choose a set bijection $\pi : P_0 \cup P_1 \longrightarrow T_0 \cup T_1$ such that $\pi(P_0) = T_0$, $\pi(P_1) = T_1$ and $\pi(p \cdot q) = \pi(p)\pi(q)$ for all $p, q \in P_0 \cup P_1$ with $p \cdot q \in P_0 \cup P_1$. Although π maps $P_0 \cup P_1$ bijectively to $T_0 \cup T_1$, we will (as before) denote elements of P_0 by v and elements of P_1 by a. We can extend π to a map from P^* to R by defining

$$\pi(a_1 \cdot a_2 \cdots a_k) = \pi(a_1)\pi(a_2) \cdots \pi(a_k).$$

Let $K_\pi = \{p \in P |\ \pi(p) = 0\}$. As before, since each path in P^* has unique initial and terminal vertices, it follows that for each $p \in P \setminus K_\pi$, there exist unique idempotents $e, f \in E$ with $\pi(p) = e\pi(p)f$. If $p = a_1 \cdot a_2 \cdots a_i$, then it follows $\pi(p) \in J^i$. By our choice of T_1, we have

$$J/J^2 = \bigoplus_{t \in T_1} (Rt + J^2)/J^2 = \bigoplus_{a \in P_1} (R\pi(a) + J^2)/J^2.$$

As in the unital case, we now explore the situation in which, for each i

$$J^i/J^{i+1} = \bigoplus_{t \in T_i} (Rt + J^{i+1})/J^{i+1}.$$

Definition VI.2. Let R be a ring with atomic set E, local units $\mathcal{U}$ and radical J. We choose T_1 as above and let P be the associated free path semigroup generated by $(T_0,\ T_1)$ with map $\pi : P^* \to R$. Let T^* be a subset of $\pi(P^*) \setminus \{0\}$, and let $T_i = \{t \in T^* | t \in J^i \setminus J^{i+1}\}$, for each $i \geq 0$. Then $(\pi,\ T^*)$ is a *binomial system* for R

if the following two conditions hold:

(B.1) $J^i/J^{i+1} = \bigoplus_{t \in T_i} (Rt + J^{i+1})/J^{i+1} = \bigoplus_{t \in T_i} (tR + J^{i+1})/J^{i+1}$

(B.2) For every $p \in P^*$ with $\pi(p) \neq 0,$ there exists an element $t_p \in T^*$ with $R\pi(p) = Rt_p$ and $\pi(p)R = t_pR.$

We say R is a *locally binomial ring* if R there exists a binomial system $(\pi,\ T^*)$, for some choice of atomic set E and T_1. As in the unital case, $T = T^* \cup \{0\}$ is an algebra semigroup. As a consequence of definition VI.2 we have the following.

Lemma VI.3. *If R is a locally binomial ring with set of local units $\mathfrak{U}$ then uRu is a unital binomial ring for all $u \in \mathfrak{U}$.*

Proof. Let $(\pi,\ T^*)$ be a binomial system for R and let $u \in \mathfrak{U}$. By definition, $T_0 = E$ is an atomic set for R. Set $U_0 = E \cap uRu \subset T_0$. Similarly, set $U_1 = uT_1u \subset T_1$. Since every $t \in T^*$ has the property $t = etf$ for some $e, f \in E$, we have $utu \in U_1$ if and only if $t = etf \in T_1$ with $e, f \in U_0$. The ring structure of R determines set maps $\sigma, \tau : T_1 \longrightarrow T_0$. Similarly, there are set maps $\sigma_u, \tau_u :\ U_1 \longrightarrow U_0$ and clearly σ_u (resp. τ_u) is the restriction of σ (resp. τ) to U_1. Hence, the free path semigroup Q generated by $(U_0,\ U_1)$ with $\sigma_Q = \sigma_u$ and $\tau_Q = \tau_u$ is a subsemigroup of the free path semigroup P. Define $\pi_u : Q \longrightarrow uRu$ as the restriction of π to Q. Define $U_i = uT_iu$ for all $i \geq 0$. Since uJu is a nilpotent ideal of uRu, for some $k \geq 0$, we have $U_i = uT_iu = 0$ for all $i \geq k+1$. Let $U^* = \cup_{i=1}^{k} U_i$. We now claim that $(\pi_u,\ U^*)$ is a binomial system for uRu.

Consider $uJ^iu/uJ^{i+1}u$. Clearly

$$\sum_{t\in U_i}(uRut + uJ^{i+1}u)/uJ^{i+1}u \subseteq uJ^iu/uJ^{i+1}u.$$

But by property **(B.1)** for $(\pi,\ T^*)$, the sum must be direct and

$$\bigoplus_{t\in U_i}(uRut + uJ^{i+1}u)/uJ^{i+1}u \subseteq uJ^iu/uJ^{i+1}u.$$

For all $t \in T_i$, if $utu \neq 0$, then $utu = t$ and $t \in U_i$. Hence

$$\begin{aligned} uJ^iu/uJ^{i+1}u = u\left(J^i/J^{i+1}\right)u \subseteq u\left(\bigoplus_{t\in T_i}(Rt + J^{i+1})/J^{i+1}\right)u \\ = \bigoplus_{t\in U_i}(uRut + uJ^{i+1}u)/uJ^{i+1}u \end{aligned}$$

and $uJ^iu/uJ^{i+1}u = \bigoplus_{t\in U_i}(uRut+uJ^{i+1}u)/uJ^{i+1u}$. We have a similar decomposition of $uJ^iu/uJ^{i+1}u$ as a right module. So $(\pi_u,\ U^*)$ satisfies property **(B.1.)** of Definition VI.2.

Since Q is a subsemigroup of the free path semigroup P, for all $p \in Q^* \subset P^*$ with $\pi_u(p) = \pi(p) \neq 0$ we have $t_p \in T^*$ with $R\pi_u(p) = Rt_p$ and $\pi_u(p)R = t_pR$. But by definition, $\pi_u(p) = e\pi_u(p)f$ for $e, f \in E \cap uRu$. Therefore, $\pi_u(p) = u\pi_u(p)u$ implying that $t_p = ut_pu$ and $t_p \in U_i$. Hence $(\pi_u,\ U^*)$ satisfies property **(B.2.)** of Definition VI.2 and $(\pi_u,\ U^*)$ is a binomial system for uRu. □

Much of what is known about artinian binomial rings extends to locally binomial rings. We now discuss the changes needed to apply Theorem IV.8 to locally binomial rings.

Definition VI.4. A locally binomial ring R with atomic set E is *cleft* if there is a semisimple atomic subring $S \subset R$ such that $R/J \cong S$ as rings and $R = S \oplus J$ as abelian groups.

As in the artinian case, for a cleft locally binomial ring with subring S, we say $(\pi,\ T^*)$ is a *strong binomial system* if for all $i \in T^*$

$$(St + J^{k+1})/J^{k+1} = (tS + J^{k+1})/J^{k+1}$$

as sets.

Note, by definition the subring S is a semisimple locally artinian ring. In our exploration of free path semigroups P and maps $\pi : P \longrightarrow R$ we made no special mention of a multiplicative identity element of R. In fact, the development of free path semigroup properties of Chapters II and IV applies to free path semigroups generated by subsets of rings with local units. Similarly, we may adapt previous results to locally binomial rings. In particular, we show Lemma III.17 extends.

Lemma VI.5. *Let R be a cleft locally binomial ring with subring S and system $(\pi,\ T^*)$. Then as a S-S bimodule,*

$$R = \bigoplus_{t \in T^*} St = \bigoplus_{t \in T^*} tS$$

Proof. Since R is cleft, it follows that as S-modules, $R = S \bigoplus J$. Because R is locally binomial, as S-modules

$$Rt + J^{i+1} = St \oplus J^{i+1}.$$

Using the binomial system of R we have as S-modules, for all $t \in T_i$

$$\begin{aligned} J^i/J^{i+1} &= \bigoplus_{t \in T_i} (Rt + J^{i+1})/J^{i+1} \\ &= \bigoplus_{t \in T_i} (St \oplus J^{i+1})/J^{i+1} \\ &= \left((\bigoplus_{t \in T_i} St) \oplus J^{i+1} \right))/J^{i+1} \end{aligned}$$

implying ${}_SJ^i = \left(\bigoplus_{t \in T_i} St \right) \oplus J^{i+1}$. Inductively, we have ${}_SR = \bigoplus_{t \in T^*} St$. A symmetric argument for right S-modules gives $R_S = \bigoplus_{t \in T^*} tS$. □

As a consequence of the above lemma, each $r \in R$ can be written as

$$r = \sum_{t_j \in T^*}^{\ell_i} t_j s_j.$$

Since the notion of a path semigroup P does not depend upon a quiver, the idea of suitable ordering $<$ that respects a binomial system holds in the setting of locally binomial rings. However, this need not be true for locally binomial rings and so we now assume it. The discussion leading up to Theorem IV.8 did not require or explicitly use an identity element for R and it is a simple check to see we may retain our previous notation and adapt $\mathcal{M}$, Γ^k, $\Gamma^k_S \otimes_S R$, and cleft pair of homomorphisms $(\delta,\ \eta)$ to the setting of locally binomial rings. Similarly, the proof of Theorem IV.8 makes no special mention of an identity element.

Then with little modification we get the following.

Theorem VI.6. *Let R be a cleft locally binomial ring with strong binomial system $(\pi,\ T^*)$ and associated free path semigroup P such that ${}_iP$ is finite for all i. Assume we have chosen a suitable ordering $<$ on P^* that respects the binomial system with $\Gamma^2 \subset P$ defined as before. For each i, $1 \leq i \leq n$ and let ${}_iR_k = {}_i\Gamma_S^k \otimes_S R$. Then each ${}_iR_k$ is a projective right R-module and there is a cleft pair of homomorphisms $(\delta,\ \eta)$ such that*

$$\begin{tikzcd}
0 & \overline{e_iR} \arrow[l] \arrow[r, bend right, "\eta_0"'] & {}_iR_0 \arrow[l, "\delta_0"'] \arrow[r, bend right, "\eta_1"'] & {}_iR_1 \arrow[l, "\delta_1"'] \arrow[r, bend right, "\eta_2"'] & {}_iR_2 \arrow[l, "\delta_2"'] \arrow[r, bend right] & \cdots \arrow[l]
\end{tikzcd}$$

is an exact sequence of right R-modules. □

Theorem VI.6 gives a bound on projective dimensions of simples over a cleft locally artininian binomial ring R. However, unlike the artinian case, this need not bound the global dimension of R. We now explore a context in which information about global dimension can be determined.

Let R be a locally artininian ring with atomic set E and set of local units $\mathcal{U} = \mathcal{U}_E$.

Lemma VI.7. *For each simple right R-module M and each $u \in \mathcal{U}$, the right uRu-module Mu is simple or zero. Conversely, for each $u \in \mathcal{U}$, if N is a simple right uRu-module, then $N \otimes_{uRu} uR$ is a simple R-module.*

Proof. An R module M simple if and only if $S \cong eR/eJ$ for some $e \in E$. For each $u \in \mathcal{U}$, either $e \in uRu$ and $(eR/eJ)u = eRu/eJu$ is a simple uRu module or $eRu \leq J$ and $(eR/eJ)u = 0$. Conversely, N is a simple uRu module if and only if

$N \cong eRu/eJu$ for some $e \in E \cap uRu$. But then

$$N \otimes_{uRu} uR \cong (eRu/eJu) \otimes_{uRu} uR \cong eR/eJ$$

is a simple R-module. $\square$

Lemma VI.8. *Let M be a right R-module and $u \in \mathcal{U}$. If Mu is projective as a uRu-module, then $M \otimes_{uRu} uR$ is a projective R-module. In particular, if for M_R we have $MuR = M$ with Mu a projective uRu-module, then M is a projective R-module.*

Proof. Consider the functors $F : \text{Mod-}uRu \longrightarrow \text{Mod-R}$ and $G : \text{Mod-}R \longrightarrow \text{Mod-uRu}$ given by $F(X) = X \otimes_{uRu} uR$ and $G(Y) = \text{Hom}_R(uR,\ Y)$, respectively. It follows that $(F,\ G)$ is an adjoint pair with G exact. For $f : M \longrightarrow M'$ we have

$$F(f) : M \otimes_{uRu} uR \longrightarrow M' \otimes_{uRu} uR$$

given by $f \otimes 1_{uR}$. Then $F(f) = 0$ implies $f \otimes 1_{uR} = 0 \Rightarrow f = 0$ and F is a faithful functor. By a well-established result (see exercise 20.8 of [3]), if M_{uRu} is projective, then $F(M)_R = M \otimes_{uRu} uR$ is projective.

For the final assertion, assume $MuR = M$ and Mu is a projective uRu-module. Clearly, $Mu \cong M \otimes_R Ru$ and $MuR \cong Mu \otimes_{uRu} uR$. If Mu is a projective uRu-module, then by the work above, $Mu \otimes_{uRu} uR \cong MuR = M$ is a projective R-module.

$\square$

Now we apply the above ideas to projective resolutions for simples over cleft locally binomial rings. Let R be a cleft locally binomial ring with subring S and strong binomial system $(\pi,\ T^*)$. Since

$$\bigcap_{i=1}^{\infty} J^i = 0$$

we have in particular, $\bigcap\limits_{i=1}^{\infty} eJ^i = 0$ for each $e \in T_0$. Note our assumption $|{}_iP| < \infty$ for all i means $|eT_0| < \infty$ for all $e \in E$. This guarantees the module eJ/eJ^2 has finite length. This leads to the following.

Lemma VI.9. *Let R be a locally binomial ring with strong binomial system $(\pi,\ T^*)$, associated free path semigroup P with suitable order $<$ that respects the binomial system. In addition, assume for all $e \in E = T_0$, eR is noetherian and we have $|eT_1| < \infty$. If M_R is finitely generated then there is some $u \in \mathfrak{U}$ such that*

$$pdim_{uRu}(Mu) \geq pdim_R M.$$

Proof. If $\mathrm{pdim}(M_R) = 0$, then M is projective implying Mu is projective for all $u \in \mathfrak{U}$ and $\mathrm{pdim}(Mu_{uRu}) = 0$. So we may assume $\mathrm{pdim}(M_R) \geq 1$ and

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] & P \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

is a short exact sequence with P_R a finitely generated projective module. This implies $\mathrm{pdim}(M_R) - 1 = \mathrm{pdim}(L_R)$. Since each $e_\alpha R$ and $P \cong \coprod\limits_{\alpha \in A} e_\alpha R$ is noetherian, both P and L are finitely generated. We can then find $u \in \mathfrak{U}$ such that $M = MuR$, $P = PuR$ and $L = LuR$. If $\mathrm{pdim}(Mu_{uRu}) = 0$, then Mu is a projective uRu-module. But $MuR = M$ implies M is a projective R-module by Lemma VI.8. So we may assume $\mathrm{pdim}(Mu_{uRu}) \geq 1$. Note that $M \otimes_R Ru \cong Mu$ for all modules M. Also, Ru is a projective left R-module, so the functor $\_\_ \otimes_R Ru$ is exact and

$$\begin{tikzcd} 0 \arrow[r] & Lu \arrow[r] & Pu \arrow[r] & Mu \arrow[r] & 0 \end{tikzcd}$$

is an exact sequence of uRu-modules. This implies $\mathrm{pdim}(Mu_{uRu}) - 1 = \mathrm{pdim}(Lu_{uRu})$ Since $L = LuR$, any minimal R-projective resolution for L

$$\begin{tikzcd} \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & L \arrow[r] & 0 \end{tikzcd}$$

has the property if $Q_i \neq 0$ then $Q_i u \neq 0$. Hence

$$\begin{tikzcd} \cdots \arrow[r] & Q_1 u \arrow[r] & Q_0 u \arrow[r] & Lu \arrow[r] & 0 \end{tikzcd}$$

is a uRu- projective resolution for Lu. This implies $\mathrm{pdim}(Lu_{uRu}) \geq \mathrm{pdim}(L_R)$. Hence

$$\mathrm{pdim}(Mu_{uRu}) - 1 = \mathrm{pdim}(Lu_{uRu}) \geq \mathrm{pdim}(L_R) = \mathrm{pdim}(M_R) - 1.$$

But this gives $\mathrm{pdim}(Mu_{uRu}) \geq \mathrm{pdim}(M_R)$.

$\square$

Theorem VI.10. *Let R be a cleft locally binomial ring with atomic set E, set of local units $\mathfrak{U} = \mathfrak{U}_E$, and strong binomial system $(\pi,\ T^*)$. Assume for all $e \in E = T_0$, eR is noetherian and we have $|eT_1| < \infty$. If gldim$(uRu) < B$ for all $u \in \mathfrak{U}$, then gldim$(R) < B$.*

Proof. First assume M is a simple right R-module and $\operatorname{gldim}(uRu) < B$ for all $u \in \mathcal{U}$. In particular $\operatorname{pdim}(N_{uRu}) < B$ for all $u \in \mathcal{U}$. For each simple R-module M, there is a $u \in \mathcal{U}$ such that $M = MuR$. Then by Lemma VI.9

$$\operatorname{pdim}(Mu_{uRu}) \geq \operatorname{pdim}(M_R).$$

But by Lemma VI.7, Mu_{uRu} is a simple module. Thus, $\operatorname{pdim}(Mu_{uRu}) < B$ and

$$B > \operatorname{pdim}(Mu_{uRu}) \geq \operatorname{pdim}(M_R).$$

Therefore, $\operatorname{pdim}(M_R) < B$ for all simple R-modules M. Now let M be a finitely generated R-module. Since M/MJ is semisimple, we have $\operatorname{pdim}(M/MJ) < B$. But since $MJ \ll M$ and R/J is semisimple, any minimal projective resolution for M/MJ is a minimal projective resolution for M. Hence $\operatorname{gldim}(R) < B$. $\square$

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