THE C*-ALGEBRAS ASSOCIATED WITH IRRATIONAL TIME HOMEOMORPHISMS OF SUSPENSIONS

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Let (X, h) be a dynamical system and let $(\widetilde{X}, \varphi)$ be the suspension flow of (X, h). Let α be a real number. In this dissertation we study the crossed product $C(\widetilde{X}) \rtimes_{\varphi_\alpha} \mathbb{Z}$ associated to the dynamical system $(\widetilde{X}, \varphi_\alpha)$. If X is finite dimensional, $K^1(X) = 0$ and φ_α is minimal and uniquely ergodic, we find the Elliott invariant of $C(\widetilde{X}) \rtimes_{\varphi_\alpha} \mathbb{Z}$. Conditions on h for the existence of α such that φ_α is minimal and uniquely ergodic are given as well as a formula to compute the topological entropy of φ_α in terms of the topological entropy of h. Several examples derived from this study are provided, including the existence of non orbit equivalent minimal dynamical systems on connected compact metric 1-dimensional spaces arising from time α maps, for the same α, of suspensions of strong orbit equivalent Cantor systems.

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INTRODUCTION

Continuous and discrete flows are two of the most fundamental concepts in topological dynamics (see e.g. [24, Chapter II]). By a continuous flow we mean a pair (X, φ) where X is a topological Hausdorff space and $\varphi\colon X \times \mathbb{R} \to X$ is a continuous map satisfying $\varphi_0 = \mathrm{Id}_X$ and $\varphi_{\alpha+\beta} = \varphi_\alpha \circ \varphi_\beta$ for all $\alpha, \beta \in \mathbb{R}$. Notice that we follow the convention of writing $\varphi_\alpha(x)$ rather than $\varphi(x, \alpha)$. A discrete flow (X, h) is a pair with X a topological Hausdorff space and $h\colon X \to X$ a homeomorphism. When the space X is compact metric, we say that the discrete flow (X, h) is a (classical) dynamical system.

Given a discrete flow (X, h) where X is a locally compact Hausdorff space, there is a group homomorphism, denoted again by h, from $\mathbb{Z}$ to the group of automorphisms $\mathrm{Aut}(C_0(X))$ of (the commutative C*-algebra) $C_0(X)$ given by

$$(h_n(f))(x) = (f \circ h^{-n})(x)$$

for $n \in \mathbb{Z}$, $f \in C_0(X)$ and $x \in X$. In other words, given a discrete flow (X, h) with X a locally compact Hausdorff space it induces an action of $\mathbb{Z}$ on the C*-algebra $C_0(X)$. In general, given a locally compact group G, a C*-algebra A and an action τ of G on A, one may construct a new C*-algebra, namely, the crossed product of A by G which is denoted by $A \rtimes_\tau G$, cf. [16]. Hence every discrete flow (X, h)

induces a crossed product $C_0(X) \rtimes_h \mathbb{Z}$. This crossed product is also known as the transformation group C*-algebra of (X, h) and sometimes denoted by $C^*(\mathbb{Z}, X, h)$. Similarly, a continuous flow (X, φ) with X a locally compact Hausdorff space induces the group homomorphism $\mathbb{R} \to \mathrm{Aut}(C_0(X))$, denoted again by φ, defined by

$$(\varphi_\alpha(f))(x) = (f \circ \varphi_{-\alpha})(x)$$

for $\alpha \in \mathbb{R}$, $f \in C_0(X)$ and $x \in X$. Hence, we may consider the crossed product $C_0(X) \rtimes_\varphi \mathbb{R}$.

When one wants to study a continuous flow, it is useful to study an associated discrete flow (see e.g. [24, Chapter I]). Conversely, one way to obtain information about a discrete flow is by studying some associated continuous flow. A standard way to construct a continuous flow from a discrete flow is by means of the so called suspension construction which we describe next, cf. [24, II.5.5]. Let (X, h) be a dynamical system. The construction could be carried out for general discrete flows; however we restrict our attention to dynamical systems for technical reasons. Consider the actions of $\mathbb{R}$ and $\mathbb{Z}$ on the space $X \times \mathbb{R}$ given by

$$((x, s), \alpha) \mapsto (x, s + \alpha)$$

and

$$((x, s), n) \mapsto (h^n(x), s - n),$$

respectively, where $(x, s) \in X \times \mathbb{R}$, $n \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$. The suspension $\widetilde{X}$ of (X, h) is the quotient of $X \times \mathbb{R}$ by the $\mathbb{Z}$-action. Let $[x, s]$ denote the image of (x, s) in $\widetilde{X}$. Then an action φ of $\mathbb{R}$ on $\widetilde{X}$ is given by $\varphi_\alpha([x,s]) = [x, s+\alpha]$ which is well defined since the $\mathbb{Z}$ and $\mathbb{R}$ actions on $X \times \mathbb{R}$ defined above commute. Then $(\widetilde{X}, \varphi)$ is a continuous flow which is called the suspension flow of (X,h), cf. [24, Lemma II.5.7]. Given a real number α we will refer to φ_α as the time α map on the suspension $\widetilde{X}$ of (X,h) or just as the time α map when the spaces involved are understood. The suspension $\widetilde{X}$ of (X,h) has the following attractive properties.

- If X is compact metric then $\widetilde{X}$ is compact metric, cf. [24, Corollary II.5.9].
- If h is minimal then $\widetilde{X}$ is connected, cf. remark on page 205 of [3].
- If X is has (covering) dimension n then $\widetilde{X}$ has dimension $n+1$, because $\widetilde{X}$ is locally the product of X with the 1-dimensional space $\mathbb{R}$.

In this thesis we study the crossed product $C(\widetilde{X}) \rtimes_{\varphi_\alpha} \mathbb{Z}$ induced by the dynamical system $(\widetilde{X}, \varphi_\alpha)$. As a motivation, observe that when h is the identity map on the one point space X, the dynamical system $(\widetilde{X}, \varphi_\alpha)$ is conjugate to (S^1, R_α) where $S^1 \cong \mathbb{R}/\mathbb{Z}$ is the unit circle and R_α is the rigid rotation $x \mapsto x + \alpha$ on S^1. Thus when α is irrational, the crossed product $C(\widetilde{X}) \rtimes_{\varphi_\alpha} \mathbb{Z}$ is isomorphic to the irrational rotation algebra A_α, cf. [23]. Therefore one of our main results, Theorem I.18, gives a different formula to compute the range of the trace on the K_0 group of irrational rotation algebras, a result originally obtained by Rieffel, Pimsner and Voiculescu [23, 21]. Here we must mention that our formula for the range of the trace on K_0 of $C(\widetilde{X}) \rtimes_{\varphi_\alpha} \mathbb{Z}$ builds upon the work of [6] and hence it works for more general (noncommutative) C*-algebras.

It has been proved by Giordano, Putnam and Skau [9, Theorem 2.1] that the (topological) orbit structure of Cantor minimal systems is related to the isomorphism class of the associated crossed products. More precisely, they showed that K-theory yields complete information about the orbit structure of these dynamical systems and combined their discovery with the fact that simple direct limits of circle algebras with real rank zero and with the same scaled ordered K-theory are necessarily isomorphic [5]. For a larger class of minimal dynamical systems, it is natural to ask whether a similar result is true provided we assume the Elliott conjecture. The Elliott conjecture states that a complete isomorphism invariant for the associated (simple) crossed products is of (ordered) K-theoretic nature. Such invariants are known as the Elliott invariants. In [10], an example is given of a Cantor minimal system and a non-homogeneous system which are not strong orbit equivalent but have the same Elliott invariants. More examples are known by now of dynamical systems which are not strong orbit equivalent but have the same Elliott invariants (for a survey, see [19]). These examples involve spaces which are not connected or have dimension at least 2. A well known result about the circle states that two minimal homeomorphism of the circle are flip conjugate if and only if their associated crossed products are isomorphic. However, dynamical systems on 1-dimensional connected compact metric spaces which are not homeomorphic to the circle have not been studied yet in this context. As a result of our study, we have succeeded in finding non orbit equivalent minimal dynamical systems on connected compact metric 1-dimensional spaces arising from time α maps, for the same α, of suspensions of strong orbit equivalent Cantor systems. We expect these dynamical systems to have the same Elliott invariants. Our example relies in the fact proved in [11] that there are two Toeplitz flows with different entropy which are simultaneously the maximal equicontinuous factors of and strong orbit equivalent to the 2-odometer. The existence of $\alpha \in \mathbb{R}$ such that the time α map on the suspension of those Toeplitz flows are minimal gives us two minimal dynamical systems on connected compact metric 1-dimensional spaces which are not orbit equivalent because they have different topological entropy. If those time α maps were in addition uniquely ergodic, our Theorem I.18 would give us that the Elliott invariants of their associated crossed products are the same. This would prove the existence of non orbit equivalent dynamical systems on connected compact metric 1-dimensional spaces having associated crossed products with the same Elliott invariants. Hence our study suggests that conditions to extend [9, Theorem 2.1] to more general minimal dynamical systems might need to take into account the entropy of the systems involved.

We have divided this work into 5 chapters. Chapter 1 will give a formula to compute the range of a trace on the K_0 group of $C(\widetilde{X}) \rtimes_{\varphi_\alpha} \mathbb{Z}$. Chapters 2 and 3 will analyze conditions to ensure the existence of a number α such that the time α map on the suspension of a dynamical system is minimal and uniquely ergodic, respectively. In Chapter 4 we will provide a formula to compute the entropy of a time

α map on the suspension of a dynamical system (X, h) in terms of the entropy of h. Finally, Chapter 5 will present several examples derived from our study, including the existence of non orbit equivalent minimal dynamical systems on connected compact metric 1-dimensional spaces arising from time α maps, for the same α, of suspensions of strong orbit equivalent Cantor systems.

We now want to explain some terminology and introduce some notation. We say that a dynamical system (x, h) (or just h) is minimal if there is no nontrivial closed h invariant subset of X or equivalently if for all x in X the orbit $\{h^n(x) \colon n \in \mathbb{Z}\}$ of x is dense in X. We say that a dynamical system (X, h) (or just h) is uniquely ergodic if there is only one h invariant Borel probability measure on X. If (X_1, h_1) and (X_2, h_2) are two dynamical systems, we say that (X_1, h_1) is conjugate to (X_2, h_2) if there exists a homeomorphism $F \colon X_1 \to X_2$ such that $F \circ h_1 = h_2 \circ F$. We say that (X_1, h_1) is flip conjugate to (X_2, h_2) if (X_1, h_1) is conjugate either to (X_2, h_2) or to (X_2, h_2^{-1}). The dynamical system (X_1, h_1) is orbit equivalent to (X_2, h_2) if there is a homeomorphism $F : X_1 \to X_2$, called an orbit map, such that $F(\{h_1^n(x) \colon n \in \mathbb{Z}\}) = \{h_2^n(F(x)) \colon n \in \mathbb{Z}\}$ for all $x \in X_1$. If $F \colon X_1 \to X_2$ is an orbit map then for each $x \in X_1$ there is an integer $n(x)$ such that $F(h_1(x)) = h_2^{n(x)}(F(x))$. Likewise there exists an integer $m(x)$ such that $F(h_1^{m(x)}(x)) = h_2(F(x))$. We call n and m the orbit cocycles associated to the orbit map F. When X_i is infinite and h_i is minimal, for $i = 1, 2$, the orbit cocycles m and n are uniquely defined integer-valued functions on X_1. (In general they are not.) Two minimal dynamical systems (X_1, h_1) and (X_2, h_2)

are strong orbit equivalent if they are orbit equivalent and there is an orbit map $F\colon X_1 \to X_2$ such that their associated orbit cocycles $m, n\colon X_1 \to \mathbb{Z}$ have at most one point of discontinuity.

If (X, h) is a minimal dynamical system then the corresponding crossed product $C(X) \rtimes_h \mathbb{Z}$ is simple. If (X, h) is a uniquely ergodic dynamical system then the corresponding crossed product $C(X) \rtimes_h \mathbb{Z}$ has a unique normalized trace.

A trace τ of a C*-algebra A is a linear map $A \to \mathbb{C}$ satisfying $\tau(ab) = \tau(ba)$ and $\tau(a^*) = \overline{\tau(a)}$ for all $a, b \in A$. A trace τ of a C*-algebra A extends to the C*-algebra $M_n(A)$ of $n \times n$ matrices with entries in A by the formula $\tau(x) = \sum_{i=1}^{n} \tau(x_{i,i})$ for all $x \in M_n(A)$. A trace τ on A induces a group homomorphism, denoted again by τ, from $K_0(A)$ to $\mathbb{R}$ by the formula $\tau([p] - [q]) = \tau(p) - \tau(q)$. A trace τ on a unital C*-algebra A is said to be normalized if $\tau(1) = 1$.

The crossed product $A \rtimes_h \mathbb{Z}$ induced by an action h of $\mathbb{Z}$ on a unital C*-algebra A is the universal C*-algebra generated by a copy of A and a unitary u satisfying $u^n a u^{-n} = h_n(a)$ for all $a \in A$ and $n \in \mathbb{Z}$. It follows that elements of the form $\sum_{n \in \mathbb{Z}} a_n u^n$ with $a_n \in A$ for each $n \in \mathbb{Z}$ satisfying $\sum_{n \in \mathbb{Z}} \|a_n\| < \infty$ form a dense subalgebra of $A \rtimes_h \mathbb{Z}$. If τ is a trace on A, invariant under the action h, it extends to a trace on the dense subalgebra of $A \rtimes_h \mathbb{Z}$ just mentioned by the formula $\sum_{n \in \mathbb{Z}} a_n u^n \mapsto \tau(a_0)$. This formula extends continuously to the whole $A \rtimes_h \mathbb{Z}$. We will denote this new trace again by τ.

If A is a separable, unital, simple C*-algebra with unique normalized trace, the Elliott invariant of A consists of three elements: the abstract group $K_1(A)$, the scaled ordered group $K_0(A)$ (with scale [1] and order defined by $x > 0$ if and only if there is $n \in \mathbb{N}$ and a projection $p \in M_n(A)$ such that $x = [p]$) and the the map $K_0(A) \to \mathbb{R}$ induced by the unique normalized trace.

Let A and B be two separable, unital simple C*-algebras. Assume that A and B have unique normalized traces τ_A and τ_B, respectively. We say that A and B have the same (or isomorphic) Elliott invariants if the groups $K_1(A)$ and $K_1(B)$ are isomorphic and there exists an (order and scale preserving) isomorphism $\varphi\colon K_0(A) \to K_0(B)$ such that $\tau_B \circ \varphi = \tau_A$.

If s is a real number we denote by $[s]$ and $\{s\}$ the integer and fractional parts of s, respectively. We will use the symbols $\mathbb{C}, \mathbb{R}, \mathbb{Q}$ and $\mathbb{Z}$ to denote the set of complex, real, rational and integer numbers, respectively. The set of complex numbers of absolute value 1 will be denoted by S^1 or by $\mathbb{T}$ when we wish to highlight the group structure of S^1. We will use the symbols $\mathbb{Z}_+$ and $\mathbb{R}_+$ to denote the set of strictly positive integers and strictly positive reals, respectively. If X is a metric space with a metric d, we will denote the (open) ball with center at x and radius ϵ by $\mathrm{B}_\epsilon(x)$, that is,

$$\mathrm{B}_\epsilon(x) = \{y \in X\colon \mathrm{d}(x, y) < \epsilon\}.$$

CHAPTER I

RANGE OF TRACE

Let (X, h) be a dynamical system and let $(\widetilde{X}, \varphi)$ be the suspension flow associated to (X, h). Fix $\alpha \in \mathbb{R}$ and assume that the time α map φ_α is minimal and uniquely ergodic. Let $\widetilde{\tau}$ be the trace of $C(\widetilde{X})$ associated to the unique invariant measure on $\widetilde{X}$. The main goal of this chapter will be to find the Elliott invariants of the crossed product $C(\widetilde{X}) \rtimes_{\varphi_\alpha} \mathbb{Z}$. The hard work will be to provide a formula for the range of $\widetilde{\tau}$ on the K_0 group of $C(\widetilde{X}) \rtimes_{\varphi_\alpha} \mathbb{Z}$. There is a number of results in the literature which give formulas to compute the range of the trace on the K_0 group of crossed products (see e.g. [20], [14] and [6]). However, we feel that the formula we provide in this chapter is more specialized and therefore it is more useful for applications. Our formula will rely on the work of [6] and so what we get is something more general than what is needed.

We begin by providing some definitions and notation, most of which are taken from [6]. Let A be a unital C*-algebra, let τ be a trace on A and let $h\colon A \to A$ be an automorphism. We say that τ is h invariant (or that h preserves τ) if $\tau(h(a)) = \tau(a)$ for all $a \in A$. An automorphism $h\colon A \to A$ on a C*-algebra A induces an automorphism on $M_n(A)$ by the formula $h(x) = (h(x_{ij}))$. We denote by $U_\infty(A)$ the usual inductive limit of the sequence of groups

$$U_1(A) \longrightarrow U_2(A) \longrightarrow \cdots \longrightarrow U_n(A) \longrightarrow U_{n+1}(A) \longrightarrow \cdots$$

where $U_n(A)$ is the subgroup of $M_n(A)$ consisting of unitaries, i.e. $U_n(A) = \{x = (x_{ij}) \in M_n(A) \colon (\overline{x_{ji}})(x_i j) = x^*x = 1 = xx^* = (x_{ij})(\overline{x_{ji}})\}$, and the map $U_n(A) \to U_{n+1}(A)$ is given by $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Definition I.1. Let A be a unital C*-algebra and let τ be a trace on A. We say that a group homomorphism

$$\det \colon U_\infty(A) \to \mathbb{T}$$

is a determinant with trace τ if for all selfadjoint $a \in M_n(A)$

$$\det(e^{ia}) = e^{i\tau(a)}.$$

Definition I.2. Let A be a unital C*-algebra and let τ be a trace on A. We say that (A, τ) is integral if $\tau(K_0(A)) \subset \mathbb{Z}$.

We know that A admits a determinant associated with τ if and only if (A, τ) is integral, cf. [6, Theorem II.10].

Definition I.3. Let A be a unital C*-algebra and let $h \colon A \to A$ be an automorphism. We denote by $K_1(A)^h$ the subgroup of fixed points for the action of h on $K_1(A)$, i.e.

$$K_1(A)^h = \{x \in K_1(A) \colon h_*(x) = x\}$$

where $h_*(x) = [h(u)]$ for $u \in M_\infty(A)$ such that $x = [u]$.

Definition I.4. Suppose that (A, τ) is an integral C*-algebra and let $h\colon A \to A$ be a trace preserving automorphism. The rotation number map of h with respect to the trace τ is the group homomorphism

$$\rho_h^\tau\colon K_1(A)^h \to \mathbb{T}$$

defined by

$$\rho_h^\tau(x) = \det(h(u^*)u)$$

for $x \in K_1(A)^h$, where $u \in U_\infty(A)$ is such that $x = [u]$ and det is a determinant for A with trace τ.

Proposition IV.2 in [6] shows that there is no ambiguity in the definition of ρ_h^τ, that is, it is in fact a well defined group homomorphism and it does not depend on the determinant used in its definition.

Definition I.5. Let A be a unital C*-algebra and let $h\colon A \to A$ be an automorphism. The suspension or mapping torus of the pair (A, h) (or just of h) is the (unital) C*-algebra

$$T_h(A) = \{f \in C(\mathbb{R}, A)\colon f(s+1) = h(f(s))\}.$$

The suspension of A is the C*-algebra

$$SA = \{f \in C([0,1], A)\colon f(0) = 0 = f(1)\}.$$

Observe that if f is in $T_h(A)$ then f restricted to $[0,1]$ gives a continuous function $[0,1] \to A$ satisfying $f(1) = h(f(0))$. Conversely, given a continuous function

$f\colon [0,1] \to A$ satisfying $f(1) = h(f(0))$, we may extend f uniquely to an element in $T_h(A)$ by the formula $f(t) = h^n(f(t-n))$ for $t \in [n, n+1]$ and $n \in \mathbb{Z}$. With this in mind, it follows that SA is the subalgebra of $T_h(A)$ consisting of all elements f in $T_h(A)$ such that $f(0) = 0$. The following result is referred as the "mapping torus exact sequence" in [7]. It is also used in [15]. We state it here as we need it in the sequel.

Lemma I.6. *Let SA and $T_h(A)$ be as above. Let $\mathrm{ev}_0\colon T_h(A) \to A$ be given by $\mathrm{ev}_0(f) = f(0)$. There is a short exact sequence*

$$0 \longrightarrow SA \xrightarrow{\ j\ } T_h(A) \xrightarrow{\mathrm{ev}_0} A \longrightarrow 0$$

where j is the inclusion map.

Proof. Given $a \in A$, the map $\mathbb{R} \ni t \mapsto th(a) + (1-t)a$ is in ${\mathrm{ev}_0}^{-1}(a)$. Hence ev_0 is onto. It remains to check that $SA = \ker(\mathrm{ev}_0)$ but this is clear by the remark above. □

Definition I.7. Let (A, τ) be a unital traced C*-algebra and let $h\colon A \to A$ be a trace preserving automorphism of A. We define a trace $\widetilde{\tau}$ on $T_h(A)$ by

$$\widetilde{\tau}(f) = \int_{[0,1]} (\tau \circ f)(t)\, d\lambda(t)$$

for $f \in T_h(A)$, where λ is the Lebesgue measure on $\mathbb{R}$.

Definition I.8. Let (A, τ) be a unital traced C*-algebra and let $h\colon A \to A$ be a trace preserving automorphism of A. Let $\widetilde{\tau}$ be as in Definition I.7 and let $\alpha \in \mathbb{R}$. We define

φ_α to be the $\widetilde{\tau}$ preserving automorphism on $T_h(A)$ given by

$$(\varphi_\alpha(f))(t) = f(t-\alpha)$$

for $f \in T_h(A)$ and $t \in \mathbb{R}$.

Lemma I.9. *Let (A, τ) be a unital traced C^*-algebra and let h be a trace preserving automorphism of A. Suppose that $(T_h(A), \widetilde{\tau})$ is integral. Let $\alpha \in \mathbb{R}$. Then*

$$\rho^{\widetilde{\tau}}_{\varphi_\alpha} = \begin{cases} \rho^{\widetilde{\tau}}_{\varphi_{\{\alpha\}}} \left(\rho^{\widetilde{\tau}}_{\varphi_1}\right)^{[\alpha]} & \text{if } [\alpha] \geq 0 \\ \rho^{\widetilde{\tau}}_{\varphi_{\{\alpha\}}} \left(\rho^{\widetilde{\tau}}_{\varphi_{-1}}\right)^{-[\alpha]} & \text{if } [\alpha] \leq 0. \end{cases}$$

Proof. Observe that $\varphi_\alpha = \varphi_{\{\alpha\}} \circ (\varphi_1)^{[\alpha]}$ if $[\alpha] \geq 0$ and $\varphi_\alpha = \varphi_{\{\alpha\}} \circ (\varphi_{-1})^{-[\alpha]}$ if $[\alpha] \leq 0$. Since the time $\{\alpha\}$, 1 and -1 maps are homotopic to the identity, their rotation number maps all have the same domain $K_1(A)$. Thus our lemma follows from IV.3 and IV.4 in [6]. □

What Lemma I.9 says is that if we want to compute the rotation number map of φ_α with respect $\widetilde{\tau}$, we only need to do it when $|\alpha| \leq 1$. We take care of this case in the next lemma.

Lemma I.10. *Under the hypothesis of previous lemma, let $p \in M_m(A)$ be a projection. Let $u \in U_m(T_h(A))$ be given by the formula*

$$u(t) = e^{2\pi i t p}$$

for $t \in [0, 1]$. Let $|\alpha| \leq 1$. Then

$$\rho^{\widetilde{\tau}}_{\varphi_\alpha}([u]) = \exp(-2\pi i \alpha \tau(p)).$$

Proof. By using the technique in the proof of Lemma I.9, we see that $\rho^{\widetilde{\tau}}_{\varphi_\alpha}\rho^{\widetilde{\tau}}_{\varphi_{-\alpha}} = 1$. Therefore it will suffice to prove the lemma for $-1 \leq \alpha \leq 0$. Consider the function $\widetilde{u} : [0,1] \to T_h(A)$ given by

$$(\widetilde{u}(s))(t) = \begin{cases} \exp\left(2\pi i[(t-1)(1-s)p + s(t-\alpha-1)p]\right) & \text{if } t-\alpha \in [0,1] \\ \exp\left(2\pi i[(t-1)(1-s)p + s(t-\alpha-1)h(p)]\right) & \text{if } t-\alpha \in [1,2] \end{cases}$$

for $s, t \in [0,1]$. For each $s \in [0,1]$, the function $\widetilde{u}(s)$ is indeed in $T_h(A)$ since it is continuous and

$$\begin{aligned} (\widetilde{u}(s))(1) &= \exp(2\pi i(-\alpha s h(p))) \\ &= h(\exp(2\pi i(-\alpha s p))) \\ &= h(\exp(2\pi i\,[(-1)(1-s)p + s(-\alpha-1)p])) \\ &= h((\widetilde{u}(s))(0)). \end{aligned}$$

Furthermore $\widetilde{u}$ is C^∞ and since

$$\varphi_\alpha(\widetilde{u}(0)) = \widetilde{u}(1)$$

we obtain that $\widetilde{u} \in T_{\varphi_\alpha}(T_h(A))$. Combining Theorems V.7 and V.12 in [6] we obtain a commutative diagram

$$\begin{array}{ccccc} K_1(T_{\varphi_\alpha}(T_h(A))) & \xrightarrow{k_1^{\varphi_\alpha}} & K_0(T_h(A) \rtimes_{\varphi_\alpha} \mathbb{Z}) & \xrightarrow{\widetilde{\tau}} & \mathbb{R} \\ & \searrow^{(\mathrm{ev}_0)_*} & \downarrow \partial & & \downarrow q \\ & & K_1(T_h(A)) & \xrightarrow{\rho^{\widetilde{\tau}}_{\varphi_\alpha}} & \mathbb{T} \end{array}$$

where q is the map $\mathbb{R} \ni t \mapsto \exp(2\pi i t) \in \mathbb{T}$ denoted by π in [6, Theorem V.12]. Since

$\mathrm{ev}_0(\widetilde{u}) = \widetilde{u}(0) = u$ we have

$$\begin{aligned} \rho^{\widetilde{\tau}}_{\varphi_\alpha}([u]) &= \rho^{\widetilde{\tau}}_{\varphi_\alpha}((\mathrm{ev}_0)_*([\widetilde{u}])) \\ &= (q \circ \widetilde{\tau} \circ k_1^{\varphi_\alpha})([\widetilde{u}]). \end{aligned}$$

We compute $\widetilde{\tau}(k_1^{\varphi_\alpha}(\widetilde{u}))$ by using [6, Theorem V.11]. First, define a function $f : [0,1] \to A$ by

$$f(t) = \begin{cases} -(t-1)p + (t-\alpha-1)p & \text{if } t-\alpha \in [0,1] \\ -(t-1)p + (t-\alpha-1)h(p) & \text{if } t-\alpha \in [1,2]. \end{cases}$$

We claim that $\widetilde{\tau}\left(\widetilde{u}'(s)^*\widetilde{u}(s)\right) = \widetilde{\tau}(f)$. Put $a(t) = (t-1)p$ and rewrite $\widetilde{u}$ as

$$(\widetilde{u}(s))(t) = \exp(2\pi i[a(t) + sf(t)]).$$

Use the power series expansion for exp to get

$$\begin{aligned} \left(\frac{d\widetilde{u}}{ds}(s)\right)(t) &= \frac{d}{ds}\sum_{k=0}^{\infty}\frac{(2\pi i)^k}{k!}[a(t)+sf(t)]^k \\ &= \sum_{k=0}^{\infty}\frac{(2\pi i)^k}{k!}\frac{d}{ds}[a(t)+sf(t)]^k \\ &= \sum_{k=1}^{\infty}\frac{(2\pi i)^k}{k!}\sum_{j=0}^{k-1}[a(t)+sf(t)]^{k-1-j}f(t)[a(t)+sf(t)]^j. \end{aligned}$$

Hence

$$
\begin{aligned}
\left(\widetilde{u}'(s)^*\widetilde{u}(s)\right)(t) &= \left(\sum_{k=1}^{\infty}\frac{(-2\pi i)^k}{k!}\sum_{j=0}^{k-1}[a(t)+sf(t)]^j f(t)[a(t)+sf(t)]^{k-1-j}\right) \\
&\qquad \cdot\left(\sum_{l=0}^{\infty}\frac{(2\pi i)^l}{l!}[a(t)+sf(t)]^l\right) \\
&= \sum_{l=1}^{\infty}\sum_{k=0}^{l-1}\frac{(-2\pi i)^{l-k}}{(l-k)!}\frac{(2\pi i)^k}{k!}\sum_{j=0}^{l-k-1}[a(t)+sf(t)]^j f(t)[a(t)+sf(t)]^{l-1-j} \\
&= \sum_{l=1}^{\infty}(2\pi i)^l\sum_{k=0}^{l-1}\frac{(-1)^{l-k}}{(l-k)!\,k!}\sum_{j=0}^{l-k-1}[a(t)+sf(t)]^j f(t)[a(t)+sf(t)]^{l-1-j} \\
&= \sum_{l=1}^{\infty}\frac{(2\pi i)^l}{l!}\sum_{k=0}^{l-1}(-1)^{l-k}\binom{l}{k}\sum_{j=0}^{l-k-1}[a(t)+sf(t)]^j f(t)[a(t)+sf(t)]^{l-1-j}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\widetilde{\tau}\left(\left(\widetilde{u}'(s)^*\widetilde{u}(s)\right)(t)\right) &= \sum_{l=1}^{\infty}\frac{(2\pi i)^l}{l!}\sum_{k=0}^{l-1}(-1)^{l-k}\binom{l}{k}(l-k)\widetilde{\tau}(f(t)[a(t)+sf(t)]^{l-1}) \\
&= \sum_{l=1}^{\infty}\frac{(2\pi i)^l}{(l-1)!}\widetilde{\tau}(f(t)[a(t)+sf(t)]^{l-1})\sum_{k=0}^{l-1}(-1)^{l-k}\binom{l-1}{k} \\
&= -2\pi i\widetilde{\tau}(f(t))
\end{aligned}
$$

as wanted. So, using the h invariance of the trace τ, we get

$$
\begin{aligned}
\widetilde{\tau}(k_1^{\varphi_\alpha}([\widetilde{u}])) &= \frac{1}{2\pi i}\int_0^1\widetilde{\tau}\left(\widetilde{u}'(s)^*\widetilde{u}(s)\right)\,ds \\
&= \frac{-2\pi i}{2\pi i}\widetilde{\tau}(f) \\
&= -\int_0^1\tau(f(t))\,dt \\
&= -\left(\int_0^{1+\alpha}\alpha\tau(p)\,dt+\int_{1+\alpha}^1\alpha\tau(p)\,dt\right) \\
&= -\alpha\tau(p).
\end{aligned}
$$

Thus $\rho_{\varphi_\alpha}^{\widetilde{\tau}}([u]) = \exp(-2\pi i\alpha\tau(p))$, as wanted. □

Proposition I.11. *Let (A, τ) be a unital traced C^*-algebra and let h be a trace preserving automorphism of A. Suppose that $(T_h(A), \widetilde{\tau})$ is integral. Let $p \in M_m(A)$ be a projection and let $u \in U_m(T_h(A))$ be given by the formula*

$$u(t) = e^{2\pi i t p}$$

with $t \in [0, 1]$. Then for all $\alpha \in \mathbb{R}$

$$\rho^{\widetilde{\tau}}_{\varphi_\alpha}([u]) = \exp(-2\pi i \alpha \tau(p)).$$

Proof. Immediate from Lemmas I.9 and I.10. ☐

Lemma I.12. *If $K_1(A) = 0$ then the inclusion $SA \xrightarrow{j} T_h(A)$ induces a surjective homomorphism $K_1(SA) \xrightarrow{j_*} K_1(T_h(A))$. In consequence, elements in $K_1(T_h(A))$ can be represented as products of unitaries of the form*

$$u(t) = e^{2\pi i t p}$$

where $p \in M_n(A)$ is a projection and $n \geq 1$.

Proof. This follows from the short exact sequence in Lemma I.6 and the Bott periodicity $K_0(A) \cong K_1(SA)$. ☐

Theorem I.13. *Let (A, τ) be a unital traced C^*-algebra and let h be a trace preserving automorphism of A. Suppose that $(T_h(A), \widetilde{\tau})$ is an integral C^*-algebra and $K_1(A) = 0$. Let $\alpha \in \mathbb{R}$. Then the following diagram is commutative*

$$\begin{array}{ccc} K_1(T_h(A)) & \xrightarrow{\rho^{\widetilde{\tau}}_{\varphi_\alpha}} & \mathbb{T} \\ {\scriptstyle k_1^h}\downarrow & & \uparrow{\scriptstyle q^\alpha} \\ K_0(A \rtimes_h \mathbb{Z}) & \xrightarrow{\tau} & \mathbb{R} \end{array}$$

where q^α denotes the map $\mathbb{R} \ni t \mapsto e^{2\pi i\alpha t} \in \mathbb{T}$ and k_1^h is the isomorphism denoted by k_1 in [6, Theorem V.3].

Proof. By Lemma I.12, it suffices to prove the theorem for elements in $K_1(T_h(A))$ represented by unitaries of the form

$$u(t) = e^{2\pi i t p}$$

where $p \in M_n(A)$ is a projection and $n \geq 1$. On the one hand Proposition I.11 says that

$$\rho^{\widetilde{\tau}}_{\varphi_\alpha}([u]) = e^{-2\pi i\alpha\tau(p)}. \tag{I.1}$$

On the other hand, using [6, Theorem V.11] we compute

$$\begin{aligned} \tau \circ k_1^h([u]) &= \frac{1}{2\pi i}\int_0^1 \tau(u'(t)^* u(t))\, dt. \\ &= \frac{1}{2\pi i}\int_0^1 \tau\left(u(t)^*(-2\pi i p)u(t)\right)\, dt \\ &= -\tau(p). \end{aligned}$$

Composing the last equation with q^α and comparing the result with I.1 completes the proof. □

Corollary I.14. *Assume the hypothesis of Theorem I.13. Then the range of the trace $\widetilde{\tau}$ on the K_0 group of $T_h(A) \rtimes_{\varphi_\alpha} \mathbb{Z}$ is*

$$\mathbb{Z} + \alpha\tau(K_0(A \rtimes_h \mathbb{Z})).$$

Proof. As φ_α is homotopic to the identity, $\rho^{\widetilde{\tau}}_{\varphi_\alpha}$ has domain the whole of $K_1(A)$. Therefore, given $x \in K_0(T_h(A) \rtimes_{\varphi_\alpha} \mathbb{Z})$, we may combine the diagrams in Theorem I.13

and [6, Theorem V.12] to obtain

$$\exp(2\pi i[\widetilde{\tau}(x) - \alpha\tau(k_1^h(\partial(x)))]) = 1. \tag{I.2}$$

Thus $\widetilde{\tau}(K_0(T_h(A) \rtimes_{\varphi_\alpha} \mathbb{Z})) \subset \mathbb{Z} + \alpha\tau(K_0(A \rtimes_h \mathbb{Z}))$. For the reverse inclusion, observe that $\mathbb{Z} \subset \widetilde{\tau}(K_0(T_h(A) \rtimes_{\varphi_\alpha} \mathbb{Z}))$ since $\widetilde{\tau}(1) = 1$. It remains to prove the inclusion $\alpha\tau(K_0(A \rtimes_h \mathbb{Z})) \subset \widetilde{\tau}(K_0(T_h(A) \rtimes_{\varphi_\alpha} \mathbb{Z}))$. For this purpose, let y be an arbitrary element in $K_0(A \rtimes_h \mathbb{Z})$. Since k_1^h is an isomorphism (cf. [6, Theorem V.3]) and $\partial\colon K_0(T_h(A) \rtimes_{\varphi_\alpha} \mathbb{Z}) \to K_1(T_h(A))$ is surjective (cf. the begining of the proof of [6, Theorem V.12]) it follows that $k_1^h \circ \partial$ is surjective. Then there is $x \in K_0(T_h(A) \rtimes_{\varphi_\alpha} \mathbb{Z})$ such that $k_1^h(\partial(x)) = y$. Using (I.2) we get that there is $k \in \mathbb{Z}$ such that $\widetilde{\tau}(x) - \alpha\tau(y) = k$ and so $\widetilde{\tau}(x - k1) = \alpha\tau(y)$. Thus $\alpha\tau(K_0(A \rtimes_h \mathbb{Z})) \subset \widetilde{\tau}(K_0(T_h(A) \rtimes_{\varphi_\alpha} \mathbb{Z}))$, as was to be proved. □

We now return to the case of commutative C*-algebras. Let (X, h) be a dynamical system and let $(\widetilde{X}, \varphi)$ be the suspension flow of (X, h). Let $\alpha \in \mathbb{R}$. Let $\pi\colon X \times \mathbb{R} \to \widetilde{X}$ be the canonical quotient map. Consider the suspension $T_{h^{-1}}(C(X))$ of $(C(X), h^{-1})$ (see Definition I.5). We will regard $T_{h^{-1}}(C(X))$ as a subalgebra of $C(X \times \mathbb{R})$. Let φ_α be the automorphism on $T_{h^{-1}}(C(X))$ as in Definition I.8. At first there seems to be a conflict in our notation, as the automorphism on $C(\widetilde{X})$ induced by the time α map $\varphi_\alpha\colon \widetilde{X} \to \widetilde{X}$ is also denoted by φ_α (and recall is defined by $(\varphi_\alpha(f))([x, s]) = f(\varphi_{-\alpha}([x, s])) = f([x, s - \alpha])$ for $f \in C(\widetilde{X})$ and $[x, s] \in \widetilde{X}$). The following results will tell us that there is virtually no distinction, either between $T_{h^{-1}}(C(X))$ and $C(\widetilde{X})$, or between the definitions of their φ_α automorphisms. We remark that this observation is known as it was noticed in [3].

Lemma I.15. *Let* (X, h), $(\widetilde{X}, \varphi_\alpha)$ *and* $\pi\colon X \times \mathbb{R} \to \widetilde{X}$ *be as above. Let* $\pi_*\colon C(\widetilde{X}) \to C(X \times \mathbb{R})$ *be the map (induced by* π*) defined by* $f \mapsto f \circ \pi$*. We have the following.*

1. $T_{h^{-1}}(C(X)) = \operatorname{Im} \pi_*$.

2. $\pi_*\colon C(\widetilde{X}) \to T_{h^{-1}}(C(X))$ *is an isomorphism.*

3. *The following diagram is commutative.*

$$\begin{array}{ccc} C(\widetilde{X}) & \xrightarrow{\varphi_\alpha} & C(\widetilde{X}) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ T_{h^{-1}}(C(X)) & \xrightarrow{\varphi_\alpha} & T_{h^{-1}}(C(X)) \end{array}$$

where the maps involved are all isomorphisms.

Proof. We have

$$\begin{aligned} f \in T_{h^{-1}}(C(X)) &\Rightarrow f(h(x), s) = f(x, s+1)\ \forall x \in X\ \forall s \in \mathbb{R} \\ &\Rightarrow g\colon \widetilde{X} \to \mathbb{C} \text{ defined by } g([x,t]) = f(x,t) \text{ is continuous} \\ &\Rightarrow f = g \circ \pi \in \operatorname{Im} \pi_* \end{aligned}$$

and

$$\begin{aligned} f \in \operatorname{Im} \pi_* &\Rightarrow \text{ there is } g \in C(\widetilde{X}) \text{ such that } f = g \circ \pi \\ &\Rightarrow f \in T_{h^{-1}}(C(X)). \end{aligned}$$

Thus (1) follows. Part (2) is a consequence of (1) and the fact that π_* is one-to-one. Part (3) follows from the following computation.

$$((\pi_* \circ \varphi_\alpha)(f))(x,t) = f([x,t-\alpha]) = \varphi_\alpha(f[x,t]) = (\varphi_\alpha \circ \pi_*(f))(x,t)$$

for $f \in C(\widetilde{X})$ and $(x,t) \in X \times \mathbb{R}$. □

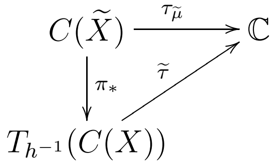
Assume now that μ is an h invariant Borel probability measure on X. Let τ_μ be the trace induced by μ on $C(X)$. Then h preserves τ_μ. As noticed at the beginning of Section 1 in [14], μ induces an $\mathbb{R}$ invariant (in particular, φ_α invariant) Borel probability measure $\widetilde{\mu}$ on $\widetilde{X}$ by the formula

$$E \mapsto \int_{X\times[0,1]} (\chi_E \circ \pi)(x,s)\, d(\mu \times \lambda)(x,s) = (\mu \times \lambda)(\pi^{-1}(E) \cap (X \times [0,1]))$$

where λ is the Lebesgue measure on $\mathbb{R}$. Let us denote by $\tau_{\widetilde{\mu}}$ the trace induced by $\widetilde{\mu}$ on $C(\widetilde{X})$. Let $\widetilde{\tau}$ be the trace on $T_{h^{-1}}(C(X))$ corresponding to Definition I.7.

Recall that we have agreed to denote again by τ_μ, $\tau_{\widetilde{\mu}}$ and $\widetilde{\tau}$ the dual traces induced by τ_μ, $\tau_{\widetilde{\mu}}$ and $\widetilde{\tau}$ on the crossed products $C(X) \rtimes_h \mathbb{Z}$, $C(\widetilde{X}) \rtimes_{\varphi_\alpha} \mathbb{Z}$ and $T_{h^{-1}}(C(X)) \rtimes_{\varphi_\alpha} \mathbb{Z}$, respectively.

Lemma I.16. *Let $\tau_{\widetilde{\mu}}$ and $\widetilde{\tau}$ be the traces described above. Then the diagram*



is commutative. In particular

1. $\tau_{\widetilde{\mu}}(K_0(C(\widetilde{X}))) = \widetilde{\tau}(K_0(T_{h^{-1}}(C(X))))$ *and*

2. $\tau_{\widetilde{\mu}}(K_0(C(\widetilde{X}) \rtimes_{\varphi_\alpha} \mathbb{Z})) = \widetilde{\tau}(K_0(T_{h^{-1}}(C(X)) \rtimes_{\varphi_\alpha} \mathbb{Z}))$.

Proof. We have that

$$\begin{aligned} \widetilde{\tau}(\pi_*(f)) &= \widetilde{\tau}(f \circ \pi) \\ &= \int_{[0,1]} \tau(f \circ \pi(\cdot, t))\, d\lambda(t) \\ &= \int_{[0,1]} \left(\int_X f \circ \pi(x,t)\, d\mu(x) \right) d\lambda(t) \\ &= \int_{X \times [0,1]} f \circ \pi(x,t)\, d(\mu \times \lambda)(x,t) \\ &= \tau_{\widetilde{\mu}}(f) \end{aligned}$$

for all $f \in C(\widetilde{X})$. Then we conclude that $\tau_{\widetilde{\mu}} = \widetilde{\tau} \circ \pi_*$ as wanted. Since by Lemma I.15 the function π_* is an isomorphism onto $T_{h^{-1}}(C(X))$, it follows that the images of $\tau_{\widetilde{\mu}}$ and $\widetilde{\tau}$ coincide. Hence the last two assertions of the lemma follow. □

Proposition I.17. *Let (X, h), $(\widetilde{X}, \varphi_\alpha)$ and $(T_{h^{-1}}(C(X)), \varphi_\alpha)$ be the dynamical systems with traces τ_μ, $\tau_{\widetilde{\mu}}$ and $\widetilde{\tau}$, as described above. Then each of the following statements implies the next.*

1. *(X, h) is minimal.*

2. *$(C(\widetilde{X}), \tau_{\widetilde{\mu}})$ is integral.*

3. *$(C(\widetilde{X}), \tau_{\widetilde{\mu}})$ and $(T_{h^{-1}}(C(X)), \widetilde{\tau})$ are integral.*

Proof. If (X,h) is minimal then $\widetilde{X}$ is connected. Therefore $(C(\widetilde{X}),\tau_{\widetilde{\mu}})$ is integral (cf. [6, Proposition VI.1]) and so $(T_{h^{-1}}(C(X)),\widetilde{\tau})$ is also integral by previous the lemma. □

We are ready to state one of the main results of this dissertation.

Theorem I.18. *Let (X,h) be a dynamical system and let α be a real number. Consider the dynamical system $(\widetilde{X},\varphi_\alpha)$ where $\widetilde{X}$ is the suspension of (X,h) and φ_α is the time α map on $\widetilde{X}$. Let μ be an h invariant Borel probability measure on X and let $\widetilde{\mu}$ be the induced φ_α invariant Borel probability measure on $\widetilde{X}$. Let τ_μ and $\tau_{\widetilde{\mu}}$ be the traces induced by μ and $\widetilde{\mu}$, respectively. Suppose that $(C(\widetilde{X}),\tau_{\widetilde{\mu}})$ is integral and $K^1(X)=0$. We have the following.*

1. *For $i=0,1$, $K_i(C(\widetilde{X})\rtimes_{\varphi_\alpha}\mathbb{Z})$ is isomorphic to*
   (a) *$K_0(C(\widetilde{X}))\oplus K_1(C(\widetilde{X}))$ and*
   (b) *$K_1(C(X)\rtimes_h\mathbb{Z})\oplus K_0(C(X)\rtimes_h\mathbb{Z})$.*
2. *The image of $\tau_{\widetilde{\mu}}$ on $K_0(C(\widetilde{X})\rtimes_{\varphi_\alpha}\mathbb{Z})$ is equal to*
$$\mathbb{Z}+\alpha\tau_\mu(K_0(C(X)\rtimes_h\mathbb{Z})).$$
3. *Furthermore, if X is finite dimensional, φ_α is both minimal and uniquely ergodic, τ_μ is injective on $K_0(C(X)\rtimes_h\mathbb{Z})$ and $\mathbb{Z}\cap\alpha\tau_\mu(K_0(C(X)\rtimes_h\mathbb{Z}))=\{0\}$ then $\tau_{\widetilde{\mu}}$ induces an order isomorphism*
$$K_0(C(\widetilde{X})\rtimes_{\varphi_\alpha}\mathbb{Z})\to\mathbb{Z}+\alpha\tau_\mu(K_0(C(X)\rtimes_h\mathbb{Z}))$$

where the order of the right hand side is inherited from $\mathbb{R}$*.*

Proof. Since φ_α is homotopic to the identity, we may use the corollary in [15] (or [17, Proposition 3.2]) and standard methods for crossed products and their K-theory to obtain

$$\begin{aligned} K_0(C(\widetilde{X}) \rtimes_{\varphi_\alpha} \mathbb{Z}) &\cong K_0(C(\widetilde{X}) \rtimes_{\mathrm{Id}} \mathbb{Z}) \\ &\cong K_0(C(\widetilde{X}) \otimes C(S^1)) \\ &\cong K_0(C(\widetilde{X})) \oplus K_1(C(\widetilde{X})). \end{aligned}$$

Furthermore, $K_i(C(\widetilde{X})) \cong K_{i-1}(C(X) \rtimes_h \mathbb{Z})$ by Lemma I.16 and the isomorphisms in [6, Theorem V.3]. Hence part (1) of the theorem follows. For part (2) we combine Corollary I.14 with Lemma I.16 to obtain

$$\begin{aligned} \tau_{\widetilde{\mu}}(K_0(C(\widetilde{X}) \rtimes_{\varphi_\alpha} \mathbb{Z})) &= \widetilde{\tau}(K_0(T_{h^{-1}}(C(X)) \rtimes_{\varphi_\alpha} \mathbb{Z})) \\ &= \mathbb{Z} + \alpha\tau_\mu(K_0(C(X) \rtimes_{h^{-1}} \mathbb{Z})) \\ &= \mathbb{Z} + \alpha\tau_\mu(K_0(C(X) \rtimes_h \mathbb{Z})) \end{aligned}$$

as wanted. Part (3) follows from [18, Theorem 4.5]. □

CHAPTER II

MINIMALITY

Let (X, h) be a dynamical system and let $(\widetilde{X}, \varphi)$ be the suspension flow associated to (X, h). Observe that the minimality of h is a necessary condition for the existence of a minimal time α map φ_α. (If M is a nontrivial closed h invariant subset of X, then $\pi(M \times \mathbb{R})$ would be a nontrivial closed φ_α invariant subset of $\widetilde{X}$.) In this chapter we show that if we require in addition to the minimality of h the minimality of h^k for some $k > 1$, then not only there exists $\alpha \in \mathbb{R}$ so that φ_α is minimal, but the set consisting of all such α is a dense G_δ set in $[0, 1]$.

Lemma II.1. *Let (X, d) be a compact metric space and let $h\colon X \to X$ be a homeomorphism. Suppose that there exists an open set $U \subset X$ and a compact set $K \subset X$ such that*

$$K \subset h(U).$$

There exists $\epsilon > 0$ such that if $g\colon X \to X$ is another homeomorphism with

$$\sup_{x \in X} \mathrm{d}(g(x), h(x)) < \epsilon,$$

then

$$K \subset g(U).$$

Proof. Let

$$\epsilon = \inf_{\substack{x \in K \\ y \in X \setminus h(U)}} \mathrm{d}(x, y).$$

Since X is metric, both K and $X \setminus h(U)$ are compact and $K \cap (X \setminus h(U)) = \emptyset$, it follows that $\epsilon > 0$. Let $g : X \to X$ be another homeomorphism for which $\sup_{x \in X} \mathrm{d}(g(x), h(x)) < \epsilon$. We must show that $K \subset g(U)$. For this purpose, let $x \in K$. The definition of ϵ implies $\mathrm{B}_\epsilon(x) \subset h(U)$ and so $gh^{-1}(\mathrm{B}_\epsilon(x)) \subset g(U)$. To complete the proof we will show that $x \in gh^{-1}(\mathrm{B}_\epsilon(x))$ or equivalently $hg^{-1}(x) \in \mathrm{B}_\epsilon(x)$. This follows because

$$\mathrm{d}(x, hg^{-1}(x)) = \mathrm{d}(g(g^{-1}(x)), h(g^{-1}(x))) \leq \sup_{x \in X} \mathrm{d}(g(x), h(x)) < \epsilon.$$

□

Lemma II.2. *Let X be a compact Hausdorff space. If $K \subset X$ is compact and U_0, U_1 are open sets in X such that*

$$K \subset U_0 \cup U_1,$$

then there are compact sets K_0 and K_1 such that

$$K = K_0 \cup K_1$$

and $K_i \subset U_i$ for $i = 0, 1$.

Proof. Put $Y_i = K \setminus U_i$. Then $Y_i \subset U_{1-i}$ for $i = 0, 1$, and $Y_0 \cap Y_1 = \emptyset$. Since X is normal, there are open sets $V_i \subset X$ such that $Y_i \subset V_i$ for $i = 0, 1$ and $V_0 \cap V_1 = \emptyset$. Then $K_i = K \setminus V_i$ is compact, $K_i \subset U_i$ for $i = 0, 1$ and $K_0 \cup K_1 = K \setminus (V_0 \cap V_1) = K$ as wanted. □

Lemma II.3. *Let X be a compact Hausdorff space. If $K \subset X$ is compact and $U_0, U_1, \ldots, U_n$ are open sets such that*

$$K \subset U_0 \cup U_1 \cup \cdots \cup U_n$$

then there are compact sets $K_0, K_1, \ldots K_n$ such that

$$K = K_0 \cup K_1 \cup \cdots \cup K_n$$

and $K_i \subset U_i$ for $i = 0, 1, \ldots n$.

Proof. By Lemma II.2, there are compact sets K_0 and K_1' such that $K = K_0 \cup K_1'$ and $K_0 \subset U_0$, $K_1' \subset U_1 \cup U_2 \cup \cdots \cup U_n$.

Using Lemma II.2 again for K_1', there are K_1 and K_2' such that $K_1' = K_1 \cup K_2'$ and $K_1 \subset U_1$, $K_2' \subset U_2 \cup U_3 \cup \cdots \cup U_n$. Repeating this argument we verify the lemma. □

Lemma II.4. *Let (X, d) be a compact metric space and let $h : X \to X$ be a homeomorphism. Suppose that there is a nonnegative integer n and an open set $U \subset X$ such that*

$$X = \bigcup_{i=0}^{n} h^i(U).$$

There exists $\epsilon > 0$ such that if $g : X \to X$ is another homeomorphism and

$$\sup_{x \in X} \mathrm{d}(g(x), h(x)) < \epsilon$$

then

$$X = \bigcup_{i=0}^{n} g^i(U).$$

Proof. To avoid triviality, assume that $n \in \mathbb{Z}_+$. Since X is compact, the sets $U, h(U), \dots, h^n(U)$ are open and $X = \bigcup_{i=0}^{n} h^i(U)$, Lemma II.3 yields existence of compact sets $K_0, K_1, \dots, K_n$ such that

$$X = \bigcup_{i=0}^{n} K_i$$

and

$$K_i \subset h^i(U)$$

for $i = 0, 1, \dots, n$. Let now $i \in \{0, 1, \dots n\}$. We use Lemma II.1 to find a positive real number ϵ_i such that if $g : X \to X$ is another homeomorphism with $\sup_{x\in X} \mathrm{d}(g(x), h^i(x)) < \epsilon_i$ then $K_i \subset g(U)$. Put $\epsilon' = \min\{\epsilon_0, \epsilon_1, \dots, \epsilon_n\}$. Furthermore, since h^i is uniformly continuous, there exists $\delta_i > 0$ such that $\mathrm{d}(h^i(x), h^i(y)) < \epsilon'/n$ whenever $\mathrm{d}(x, y) < \delta_i$.

Let $\epsilon = \min\{\epsilon'/n, \delta_2, \delta_3, \dots, \delta_n\}$. Let $g\colon X \to X$ be another homeomorphism with $\sup_{x\in X} \mathrm{d}(g(x), h(x)) < \epsilon$. To show that $X = \bigcup_{i=0}^{n} g^i(U)$, it will suffice to prove $K_i \subset g^i(U)$ for each $i = 0, 1, \dots, n$. We have $K_0 \subset U$ by construction. We prove by induction on $1 < i \leq n$ that $\sup_{x\in X} \mathrm{d}(g^i(x), h^i(x)) < i\epsilon'/n$. When $i = 1$ we have $\sup_{x\in X} \mathrm{d}(g(x), h(x)) < \epsilon \leq \epsilon'/n$. Let $i \in \{2, 3, \dots, n\}$. Assume that $\sup_{x\in X} \mathrm{d}(g^{i-1}(x), h^{i-1}(x)) < (i-1)\epsilon'/n$. Let $x_0 \in X$. Since

$$\mathrm{d}(g(x_0), h(x_0)) \leq \sup_{x\in X} \mathrm{d}(g(x), h(x)) < \epsilon \leq \delta_{i-1},$$

the uniform continuity of h^{i-1} gives $\mathrm{d}(h^{i-1}(g(x_0)), h^{i-1}(h(x_0))) < \epsilon'/n$. So

$$\begin{aligned} \mathrm{d}(g^i(x_0), h^i(x_0)) &\leq \mathrm{d}(g^{i-1}(g(x_0)), h^{i-1}(g(x_0))) + \mathrm{d}(h^{i-1}(g(x_0)), h^{i-1}(h(x_0))) \\ &< \sup_{x \in X} \mathrm{d}(g^{i-1}(x), h^{i-1}(x)) + \epsilon'/n \\ &< i\epsilon'/n. \end{aligned}$$

Thus $\sup_{x \in X} \mathrm{d}(g^i(x), h^i(x)) < i\epsilon'/n \leq \epsilon_i$ and so $K_i \subset g^i(U)$, as was to be proved. □

Lemma II.5. *Let X be a compact metric space and let $\{U_i\}_{i \in \mathbb{Z}_+}$ be a countable basis for the topology of X. The set*

$$\{h \in \mathrm{Homeo}(X) \colon h \textit{ is minimal}\}$$

is equal to

$$\bigcap_{i \geq 1} \left\{ h \in \mathrm{Homeo}(X) \colon \textit{there exists } n \in \mathbb{Z}_+ \textit{ such that } X = \textstyle\bigcup_{j=0}^{n} h^j(U_i) \right\}.$$

Proof. The proof of Proposition 4.1 in [8] shows that the set

$$\{h \in \mathrm{Homeo}(X) \colon h \text{ is minimal}\}$$

is equal to

$$\bigcap_{i \geq 1} \omega_{U_i}$$

where

$$\omega_{U_i} = \left\{ f \in \mathrm{Homeo}(X) \colon \text{there exists } n \in \mathbb{Z}_+ \text{ such that } X = \textstyle\bigcup_{j=0}^{n} h^j(U_i) \right\}$$

The lemma follows. □

Definition II.6. Let (X, φ) be a continuous flow where X is a compact metric space. For $U \subset X$ open define the set $V(U) \subset [0,1]$ by

$$V(U) = \left\{ \alpha \in [0,1] \colon \text{there exists } n \in \mathbb{Z}_+ \text{ such that } X = \textstyle\bigcup_{j=0}^{n} \varphi_\alpha^j(U) \right\}.$$

Lemma II.7. *Let (X, φ) be a continuous flow where X is a compact metric space. Let $\{U_i\}_{i \in \mathbb{Z}_+}$ be a countable basis for the topology of X. The set*

$$\{\alpha \in [0,1] \colon \varphi_\alpha \textit{ is minimal}\}$$

is equal to

$$\bigcap_{i \geq 1} V(U_i)$$

where $V(U_i)$ is as in Definition II.6.

Proof. Let α be a number in $[0,1]$ such that φ_α is minimal. Let $i \geq 1$. By Lemma II.5 we obtain

$$\varphi_\alpha \in \left\{ h \in \mathrm{Homeo}(X) \colon \text{there exists } n \in \mathbb{Z}_+ \text{ such that } X = \textstyle\bigcup_{j=0}^{n} h^j(U_i) \right\}.$$

So there exists $n \in \mathbb{Z}_+$ such that $X = \displaystyle\bigcup_{j=0}^{n} \varphi_\alpha^i(U_i)$. Hence $\alpha \in V(U_i)$. This proves the inclusion $\{\alpha \in [0,1] \colon \varphi_\alpha \text{ is minimal}\} \subset \displaystyle\bigcap_{i \geq 1} V(U_i)$.

To prove the opposite inclusion, let $\alpha \in \displaystyle\bigcap_{i \geq 1} V(U_i)$. Let U be an open set in X. Then there exists $i \geq 1$ such that $U_i \subset U$. Since $\alpha \in V(U_i)$, there is $n \in \mathbb{Z}_+$ such that $X = \displaystyle\bigcup_{j=0}^{n} \varphi_\alpha^j(U_i)$. So we get $X = \displaystyle\bigcup_{j=-\infty}^{\infty} \varphi_\alpha^j(U)$. By [25, Theorem 5.1] we conclude that φ_α is minimal, as wanted. ☐

Lemma II.8. *The set $V(U)$ of Definition II.6 is open.*

Proof. Put

$$T = \left\{\alpha \in \mathbb{R}\colon \text{there exists } n \in \mathbb{Z}_+ \text{ such that } X = \bigcup_{j=0}^{n} \varphi_\alpha^j(U)\right\}.$$

Observe that

$$V(U) = T \cap [0,1].$$

Hence it is equivalent to show that T is open in $\mathbb{R}$. Let $\alpha \in T$. Then there exists $n \in \mathbb{Z}_+$ such that $X = \bigcup_{j=0}^{n} \varphi_\alpha^j(U)$. Let d be a metric on X. By Lemma II.4, there exists $\epsilon > 0$ such that if $g : X \to X$ is another homeomorphism and $\sup_{x\in X} \mathrm{d}(g(x), \varphi_\alpha(x)) < \epsilon$ then $X = \bigcup_{j=0}^{n} g^j(U)$. Since the restriction of φ to $X \times [\alpha - \epsilon, \alpha + \epsilon]$ is uniformly continuous, there exists $0 < \delta < 1$ such that $\mathrm{d}(\varphi_s(x), \varphi_t(y)) < \epsilon/2$ whenever $\max\{\mathrm{d}(x, y), |s-t|\} < \delta$ with $s, t \in [\alpha - \epsilon, \alpha + \epsilon]$. Put $\epsilon' = \min\{\delta, \epsilon\}$. We claim that $\mathrm{B}_{\epsilon'}(\alpha) \subset T$. Indeed, given $x \in X$ we have

$$\begin{aligned} \beta \in \mathrm{B}_{\epsilon'}(\alpha) \;&\Rightarrow\; |\alpha - \beta| < \epsilon' \\ &\Rightarrow\; \max\{\mathrm{d}(x, x),\, |\alpha - \beta|\} = |\alpha - \beta| < \epsilon' \le \delta \text{ and } \alpha, \beta \in [\alpha - \epsilon, \alpha + \epsilon] \\ &\Rightarrow\; \mathrm{d}(\varphi_\alpha(x),\, \varphi_\beta(x)) < \epsilon/2. \end{aligned}$$

We have proved that if $\beta \in \mathrm{B}_{\epsilon'}(\alpha)$ then $\sup_{x\in X} \mathrm{d}(\varphi_\alpha(x), \varphi_\beta(x)) < \epsilon$ and so $X = \bigcup_{j=0}^{n} \varphi_\beta^j(U)$. This proves that $\beta \in T$. Hence $\mathrm{B}_{\epsilon'}(\alpha) \subset T$, as wanted. ☐

Lemma II.9. *Let $N > 0$ and let P be a set consisting of infinitely many powers of a* *single prime number. Then the set*

$$Q = \{\tfrac{p}{q} \colon p \in P,\ q > N \ \textit{and}\ (p,q) = 1\} \cap [0,1]$$

is dense in $[0,1]$.

Proof. Because $\mathbb{Q} \cap (0,1)$ is dense in $[0,1]$, it will suffice to show that Q is dense in $\mathbb{Q} \cap (0,1)$. Write $P = \{p_1, p_2, \ldots, p_n, p_{n+1}, \ldots\}$ with $p_n < p_{n+1}$ for $n \in \mathbb{Z}_+$. Let $\epsilon > 0$. Let $r \in \mathbb{Q} \cap (0,1)$. We will complete the proof once we find $\frac{p_k}{q} \in Q$ such that $\left|\frac{p_k}{q} - r\right| < \epsilon$. Let

$$x_n = \begin{cases} \dfrac{p_n + 1}{r} & \text{if } \left(p_n, \left[\frac{p_n+1}{r}\right]\right) = 1, \\ \dfrac{p_n + 1}{r} + 1 & \text{otherwise} \end{cases}$$

for $n \in \mathbb{Z}_+$. Then $\{x_n\}_{n \in \mathbb{Z}_+}$ is a strictly increasing sequence of positive real numbers since $p_n + r < p_{n+1}$ for all $n \in \mathbb{Z}_+$. Furthermore, $(p_n, [x_n]) = 1$ for all $n \geq 1$ because the p_n's are powers of a single prime. Then for each $n \in \mathbb{Z}_+$ we have

$$\begin{aligned} x_n - 1 \leq [x_n] \leq x_n \quad &\Rightarrow \quad \tfrac{p_n+1}{r} - 1 \leq [x_n] \leq \tfrac{p_n+1}{r} + 1 \\ &\Rightarrow \quad \tfrac{r}{1+(1+r)/p_n} \leq \tfrac{p_n}{[x_n]} \leq \tfrac{r}{1+(1-r)/p_n}. \end{aligned}$$

Since the elements of P form a strictly increasing sequence, we obtain that the strictly monotone sequences of real numbers at the ends of the last inequality both converge to r. Hence there exists k such that $r(N+1) - 1 < p_k$ and $\left|\frac{p_k}{[x_k]} - r\right| < \epsilon$. Put $q = [x_k]$ and we are done. $\square$

Proposition II.10. *Let (X, h) be a minimal dynamical system and let $k > 1$ be an integer such that h^k is also minimal. If p is a prime divisor of k then h^{p^n} is minimal for all $n \geq 0$.*

Proof. Let p be a prime divisor of k. Then h^p is minimal. (If M is h^p invariant then M is h^k invariant.) Let $n > 1$. We claim that h^{p^n} is minimal. Indeed, if h^{p^n} is not minimal, we use [24, II.9.6(7)] to find a clopen set $Y \subset X$ and a divisor l of p^n such that X is the disjoint union of $Y, h(Y), h^2(Y), \ldots, h^{l-1}(Y)$ and $h^l(Y) = Y$. Note that $l > 1$ since $Y \subset X$ is nontrivial because h^{p^n} is not minimal. As p must divide l, the set $Y \cup h^p(Y) \cup h^{2p}(Y) \cup \cdots \cup h^{l-p}(Y)$ is nontrivial, closed, and h^p invariant, contradicting the minimality of h^p. $\square$

Lemma II.11. *Let (X, h) be a dynamical system and let $(\widetilde{X}, \varphi)$ be the suspension flow of (X, h). Let $U \subset \widetilde{X}$ be open. Consider the set $V(U)$ as defined in Definition II.6. If h and h^k are minimal where $k > 1$ is some integer then $V(U)$ is dense in $[0, 1]$.*

Proof. Choose an open set $A \subset X$ and an interval $I = (a, b) \subset \mathbb{R}$ such that $W = \pi(A \times I) \subset U$, where $\pi : X \times \mathbb{R} \to \widetilde{X}$ is the canonical quotient map. We may regard A as an element of some countable basis for the topology of X. It is clear that $V(W) \subset V(U)$. To prove the lemma we will show that $V(W)$ is dense.

By Proposition II.10 there is a set P containing infinitely many powers of a single prime such that h^p is minimal for all $p \in P$. Let $N > \dfrac{1}{b-a} > 0$ be an integer. By Lemma II.9, it suffices to show that if $p \in P$ and $q > N$ satisfies $(p, q) = 1$ and

$\frac{p}{q} \in [0,1]$ then $\frac{p}{q} \in V(W)$.

Let $p \in P$ and let $q > N$ such that $(p,q) = 1$ and $\frac{p}{q} \in [0,1]$. Since h^p is minimal and A is an element of a countable basis, Lemma II.5 gives the existence of some $m \in \mathbb{Z}_+$ such that $\bigcup_{i=0}^{m} h^{ip}(A) = X$. Let $\mathcal{I}$ be the image of $I = (a,b)$ under the canonical quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. Let $R_{p/q}$ denote the rigid rotation $x \mapsto x + p/q$ on $\mathbb{R}/\mathbb{Z}$. Then $\mathcal{I} \cup R_{p/q}(\mathcal{I}) \cup \cdots \cup R_{p/q}^{q-1}(\mathcal{I}) = \mathbb{R}/\mathbb{Z}$ since $q > N > \frac{1}{b-a}$ and $(p,q) = 1$. Let $n = mq + q - 1$. We claim that $\widetilde{X} = \bigcup_{j=0}^{n} \varphi_{p/q}^{j}(W)$. Indeed, let $[x,s] \in \widetilde{X}$ with $0 \le s < 1$. Then $s \in R_{p/q}^{k}(\mathcal{I})$ for some $k \in \{0,1,\ldots,q-1\}$. So $s = t + kp/q - [t + kp/q]$ for some $t \in I$. Pick $i \in \{0,1,\ldots,m\}$ such that $h^{-[t+kp/q]}(x) \in h^{ip}(A)$. Then $y = h^{-[t+kp/q]-ip}(x) \in A$.

Hence

$$
\begin{aligned}
[x,s] &= \pi(x,s) \\
&= \pi\left(h^{-[t+kp/q]-ip}(x), s + [t+kp/q] + ip\right) \\
&= \pi\left(y, s + [t+kp/q] + ip\right) \\
&= \pi\left(y, t + kp/q - [t+kp/q] + [t+kp/q] + ip\right) \\
&= \pi\left(y, t + kp/q + ip\right) \\
&= \pi\left(y, t + (k+iq)p/q\right) \\
&= \varphi_{p/q}^{k+iq}(\pi(y,t)) \\
&\in \bigcup_{j=0}^{n} \varphi_{p/q}^{j}\left(W\right).
\end{aligned}
$$

□

Proposition II.12. *Let (X, h) be a minimal dynamical system and let $(\widetilde{X}, \varphi)$ be the suspension flow of (X, h). Assume that h^k is minimal for some $k > 1$. Then the set*

$$\{\alpha \in [0,1] \colon \varphi_\alpha \text{ is minimal}\}$$

is a dense G_δ set.

Proof. Let $\{U_i\}_{i \in \mathbb{Z}_+}$ be a countable basis for the topology of $\widetilde{X}$. By Lemma II.7, the set

$$\{\alpha \in [0,1] \colon \varphi_\alpha \text{ is minimal}\}$$

is equal to

$$\bigcap_{i \geq 1} V(U_i)$$

where $V(U_i)$ is as in Definition II.6. But, for each $i \geq 1$, the set $V(U_i)$ is open by Lemma II.8 and dense by Lemma II.11. The proposition now follows. □

CHAPTER III

UNIQUE ERGODICITY

Let (X,h) be a dynamical system and let $(\widetilde{X},\varphi)$ be the suspension flow associated to (X,h). Observe that the unique ergodicity of h is a necessary condition for the existence of a uniquely ergodic time α map φ_α. (If μ and ν are two different h invariant measures on X then, following the remark before Lemma I.16, we may construct two different φ_α invariant measures $\widetilde{\mu}$ and $\widetilde{\nu}$ on $\widetilde{X}$.) It was very pleasant to discover that, just as for minimality (see Chapter II), if we require in addition to the unique ergodicity of h the unique ergodicity of h^k for some $k>1$, then not only does there exist $\alpha \in \mathbb{R}$ such that φ_α is uniquely ergodic, but the set of such α is a dense G_δ set in $[0,1]$.

Lemma III.1. *Let X be a compact metric space, let $\{f_i\}_{i\in\mathbb{Z}_+}$ be a dense set in $C(X)$ and let $x_0 \in X$ be fixed. The set*

$$\{h \in \mathrm{Homeo}(X)\colon h \textit{ is uniquely ergodic}\}$$

is equal to

$$\bigcap_{i\geq 1}\left\{h \in \mathrm{Homeo}(X)\colon \inf\nolimits_{n\geq 1}\left\|\tfrac{1}{n}\sum\nolimits_{k=0}^{n-1}\left(f_i\circ h^k - f_i\circ h^k(x_0)\right)\right\| = 0\right\}.$$

Proof. The proof of Proposition 4.4 in [8] shows that the set

$$\{h \in \mathrm{Homeo}(X)\colon h \text{ is uniquely ergodic}\}$$

is equal to

$$\bigcap_{i\geq 1} E_i^{-1}(0),$$

where $E_i\colon \operatorname{Homeo}(X)\to\mathbb{R}_+$ is the function defined by

$$E_i(h) = \inf_{n\geq 1} \left\| \tfrac{1}{n}\sum_{k=0}^{n-1}\left(f_i\circ h^k - f_i\circ h^k(x_0)\right)\right\|.$$

Hence

$$\begin{aligned} E_i^{-1}(0) &= \{h\in\operatorname{Homeo}(X)\colon E_i(h)=0\} \\ &= \left\{h\in\operatorname{Homeo}(X)\colon \inf_{n\geq 1}\left\|\tfrac{1}{n}\sum_{k=0}^{n-1}\left(f_i\circ h^k - f_i\circ h^k(x_0)\right)\right\| = 0\right\}. \end{aligned}$$

The lemma follows. □

Definition III.2. Let (X,φ) be a continuous flow where X is a compact metric space. For $f\in C(X)$ and $\epsilon>0$, define the set $V(f,\epsilon)\subset[0,1]$ by

$$V(f,\epsilon) = \left\{\alpha\in[0,1]\colon \exists n\in\mathbb{Z}_+\ \exists c\in\mathbb{C} \text{ such that } \left\|\tfrac{1}{n}\sum_{k=0}^{n-1} f\circ\varphi_\alpha^k - c\right\| < \epsilon\right\}.$$

Lemma III.3. *Let (X,φ) be a continuous flow where X is a compact metric space. Let $\{f_i\}_{i\in\mathbb{Z}_+}$ be a dense set in $C(X)$. The set*

$$\{\alpha\in[0,1]\colon \varphi_\alpha \text{ is uniquely ergodic}\}$$

is equal to

$$\bigcap_{i\geq 1}\bigcap_{j\geq 1} V(f_i,\tfrac{1}{j})$$

where $V(f_i,\frac{1}{j})$ is as in Definition III.2.

Proof. Let α be a number in $[0,1]$ such that φ_α is uniquely ergodic. Let $i \geq 1$ and let $j \geq 1$. Fix $x_0 \in X$. By Lemma III.1, there exists $n > 0$ such that

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} \left(f_i \circ \varphi_\alpha^k - f_i \circ \varphi_\alpha^k(x_0) \right) \right\| < \frac{1}{j}.$$

Put $c = \dfrac{1}{n} \displaystyle\sum_{k=0}^{n-1} f_i \circ \varphi_\alpha^k(x_0)$. Then $\alpha \in V\left(f_i, \frac{1}{j}\right)$. This proves the inclusion $\{\alpha \in [0,1] \colon \varphi_\alpha \text{ is uniquely ergodic}\} \subset \displaystyle\bigcap_{i\geq 1} \bigcap_{j \geq 1} V\left(f_i, \frac{1}{j}\right)$.

For the opposite inclusion, let $\alpha \in \displaystyle\bigcap_{i\geq 1} \bigcap_{j \geq 1} V\left(f_i, \frac{1}{j}\right)$, let $i \geq 1$ and let $\epsilon > 0$. Choose j such that $\dfrac{1}{j} < \dfrac{\epsilon}{2}$. Since $\alpha \in V\left(f_i, \frac{1}{j}\right)$, there is $N > 0$ and $c \in \mathbb{C}$ such that

$$\left\| \frac{1}{N} \sum_{k=0}^{N-1} f_i \circ \varphi_\alpha^k - c \right\| < \frac{\epsilon}{2}.$$

In particular, for x_0 as in Lemma III.1,

$$\left| \frac{1}{N} \sum_{k=0}^{N-1} f_i \circ \varphi_\alpha^k(x_0) - c \right| < \frac{\epsilon}{2}.$$

Thus

$$\left\| \frac{1}{N} \sum_{k=0}^{N-1} \left(f_i \circ \varphi_\alpha^k - f_i \circ \varphi_\alpha^k(x_0) \right) \right\| < \epsilon.$$

Therefore

$$\inf_{n \geq 1} \left\| \frac{1}{n} \sum_{k=0}^{n-1} \left(f_i \circ \varphi_\alpha^k - f_i \circ \varphi_\alpha^k(x_0) \right) \right\| = 0.$$

Since i was chosen arbitrarily, Lemma III.1 yields unique ergodicity of φ_α. □

Lemma III.4. *The set $V(f, \epsilon)$ of Definition III.2 is open in* $[0,1]$.

Proof. Put

$$T = \left\{ \alpha \in \mathbb{R} \colon \exists n \in \mathbb{Z}_+ \ \exists c \in \mathbb{C} \text{ such that } \left\| \frac{1}{n} \sum_{j=0}^{n-1} f \circ \varphi_\alpha^j - c \right\| < \epsilon \right\}.$$

Observe that

$$V(f,\epsilon) = T \cap [0,1].$$

Hence it is equivalent to show that T is open in $\mathbb{R}$. For this purpose, let $\{\alpha_k\}$ be a sequence of real numbers in $\mathbb{R} \setminus T$ such that $\alpha_k \to \alpha$ as $k \to \infty$. Let $n \in \mathbb{Z}_+$ and let $c \in \mathbb{C}$. Then for all k, $\left\| \frac{1}{n} \sum_{j=0}^{n-1} f \circ \varphi_{\alpha_k}^{j} - c \right\| \geq \epsilon$. Taking limits as $k \to \infty$, we obtain $\alpha \in \mathbb{R} \setminus T$, as was to be proved. □

The following lemma is the analog of [24, II.9.6(7)] for the uniquely ergodic setting. We refer the reader to [25] for an explanation of the terminology and notation used in the proof and not explained here.

Lemma III.5. *Let (X,h) be a uniquely ergodic dynamical system with unique h invariant probability measure μ. Then for all $k \in \mathbb{Z}_+$ there is a divisor l of k and a measurable set $E \subset X$ such that $\mu(E) = \frac{1}{l}$, $h^l(E) = E$ and $E \cup h(E) \cup \cdots \cup h^{l-1}(E)$ is μ almost equal to X.*

Proof. Let $k \in \mathbb{Z}_+$. Let ν be an ergodic h^k invariant measure. The set

$$G = \{n \in \mathbb{Z} \colon h^n(\nu) = \nu\}$$

is a group and G contains k by assumption. Let $l > 0$ be a generator of G. Then l divides k. Since μ is the only h invariant measure, we have

$$\mu = \frac{1}{l}\left(\nu + h(\nu) + h^2(\nu) + \cdots + h^{l-1}(\nu)\right). \tag{III.1}$$

Furthermore, for $i = 1, 2, \ldots, l-1$ it follows that $h^i(\nu)$ is ergodic and $\nu \neq h^i(\nu)$. Hence for $i = 1, 2, \ldots, l-1$, ν is mutually singular with respect to $h^i(\nu)$, and so there is a measurable set $F_i \subset X$ such that $\nu(F_i) = 1 = (h^i(\nu))(X \setminus F_i)$. Put $E_i = \bigcap_{j=-\infty}^{\infty} h^{jl}(F_i)$. Then $h^l(E_i) = E_i$ and $\nu(E_i) = 1 = (h^i(\nu))(X \setminus E_i)$. Set $E = \bigcap_{i=0}^{l-1} E_i$. It follows from III.1 that $\mu(E) = \frac{1}{l}$ and since $h^l(E) = E$ we also have

$$\mu\left(\bigcup_{j=0}^{\infty} h^j(E)\right) = \mu\left(\bigcup_{j=0}^{l-1} h^j(E)\right). \tag{III.2}$$

But since $\mu(E) > 0$ and μ is ergodic, the left hand side of III.2 is equal to one. The proof is now complete. □

We remark that the proof of Lemma III.5 shows that l can be chosen to be strictly greater than 1 when h^k is not uniquely ergodic. Indeed, if h^k is not uniquely ergodic one may choose ν an ergodic h^k invariant measure different from μ. Then equation (III.1) forces $l > 1$.

Proposition III.6. *Let (X, h) be a uniquely ergodic dynamical system and let $k > 1$ be an integer such that h^k is also uniquely ergodic. If p is a prime divisor of k then h^{p^n} is uniquely ergodic for all $n \geq 0$.*

Proof. Let p be a prime divisor of k. Since the set of h^p invariant probability measures is contained in the set of h^k invariant probability measures and h^k is uniquely ergodic, we conclude that h^p is uniquely ergodic. Let μ be the unique h invariant measure. Then μ is also the unique h^p invariant measure. Let $n > 1$. We claim that h^{p^n} is uniquely ergodic. Indeed, if h^{p^n} is not uniquely ergodic, we use Lemma III.5 to find a divisor l of p^n and a measurable set $E \subset X$ such that $h^{p^n}(E) = E$, E has μ measure $\frac{1}{l}$ and $E \cup h(E) \cup h^2(E) \cup \cdots \cup h^{l-1}(E)$ is μ almost equal to X. Note $l > 1$ by the remark above since h^{p^n} is not uniquely ergodic. As p must divide l, the set $E \cup h^p(E) \cup h^{2p}(E) \cup \cdots \cup h^{l-p}(E)$ is h^p invariant and has measure in $(0, 1)$, contradicting the ergodicity of h^p with respect μ. Thus h^{p^n} is uniquely ergodic and we are done. □

Lemma III.7. *Let (X, h) be a dynamical system and let $(\widetilde{X}, \varphi)$ be the suspension flow of (X, h). Let $f \in C(\widetilde{X})$ and let $\epsilon > 0$. Consider the set $V(f, \epsilon)$ as defined in Definition III.2. If h and h^k are uniquely ergodic where $k > 1$ is some integer then $V(f, \epsilon)$ is dense in $[0, 1]$.*

Proof. Let d be a metric on $\widetilde{X}$ and let d′ be a metric on X. Since f is uniformly continuous, there is $\delta > 0$ such that $|f([x, s]) - f([y, t])| < \dfrac{\epsilon}{2}$ whenever $\mathrm{d}([x, s], [y, t]) < \delta$. Since $\pi|_{X \times [0,1]}$ is also uniformly continuous, there is δ_1 such that $\mathrm{d}([x, s], [y, t]) < \delta$ whenever $\max\{\mathrm{d}'(x, y), |s - t|\} < \delta_1$ with $0 \le s, t < 1$.

By Proposition III.6 there is a set P containing infinitely many powers of a single prime such that h^p is uniquely ergodic for all $p \in P$. Let $N > \dfrac{1}{\delta_1}$. By Lemma II.9 it suffices to show that if $p \in P$ and $q > N$ satisfies $(p, q) = 1$ and $\dfrac{p}{q} \in [0, 1]$ then $\dfrac{p}{q} \in V(f, \epsilon)$.

Let $p \in P$ and let $q > N$ such that $(p, q) = 1$ and $\dfrac{p}{q} \in [0, 1]$. Define functions

$T_m : X \to \mathbb{C}$ by

$$\begin{aligned} T_m(x) &= f([x, mp/q]) \\ &= f \circ \varphi_{p/q}^m([x, 0]) \end{aligned}$$

for $m \in \mathbb{Z}$. Since f is continuous, each T_m is continuous. Since h^p is uniquely ergodic, the sequence $\frac{1}{n}\sum_{k=0}^{n-1} T_m \circ h^{kp}$ converges uniformly as $n \to \infty$ to a constant $c_m = \int_X T_m \, d\mu$, cf. [25, Theorem 6.19]. For all nonnegative integers k and i it is easy to see that

$$T_{kq+i} = T_i \circ h^{kp}.$$

The h^p invariance of μ then gives $c_{q+i} = c_i$ for all integer $i \geq 0$. Put $c = \frac{1}{q}(c_0 + c_1 + \cdots + c_{q-1})$. Let $i, r \in \{0, 1, \ldots, q-1\}$. Since

$$\frac{1}{nq}\sum_{j=0}^{qn-1} T_{i+j} = \frac{1}{q}\sum_{j=0}^{q-1}\left(\frac{1}{n}\sum_{k=0}^{n-1} T_{i+j} \circ h^{kp}\right)$$

and for $r \geq 1$

$$\begin{aligned} \frac{1}{nq+r}\sum_{j=0}^{qn+r-1} T_{i+j} = & \frac{n+1}{nq+r}\sum_{j=0}^{r-1}\left(\frac{1}{n+1}\sum_{k=0}^{n} T_{i+j} \circ h^{kp}\right) \\ & + \frac{n}{nq+r}\sum_{j=r}^{q-1}\left(\frac{1}{n}\sum_{k=0}^{n-1} T_{i+j} \circ h^{kp}\right), \end{aligned}$$

we see that the sequence $\frac{1}{nq+r}\sum_{j=1}^{nq+r-1} T_{i+j}$ converges uniformly as $n \to \infty$ to c. We claim that this implies the uniform convergence of the sequence $\frac{1}{n}\sum_{j=0}^{n-1} T_{i+j}$ as $n \to \infty$ to the same constant c. Indeed, let $\epsilon' > 0$. For $r = 0, 1, \ldots, q-1$, choose $N_r' > 0$ such that

$$\left\| \frac{1}{nq+r} \sum_{j=0}^{nq+r-1} T_{i+j} - c \right\| < \epsilon'$$

whenever $nq + r > N'_r$. Put $N' = \max\{N'_r : r = 0, 1, \ldots, q-1\}$. If $m > N'$, write $m = nq + r$ with $r \in \{0, 1, \ldots, q-1\}$. Then $m = nq + r > N' \geq N'_r$ gives

$$\left\| \frac{1}{m} \sum_{j=0}^{m-1} T_{i+j} - c \right\| < \epsilon'.$$

This completes the proof of the claim. Let now $i \in \{0, 1, \ldots, q-1\}$. Using the claim we just proved, we may choose $N_i > 0$ such that for all $n > N_i$

$$\left\| \frac{1}{n} \sum_{j=0}^{n-1} T_{i+j} - c \right\| < \frac{\epsilon}{2}.$$

Let $N = \max\{N_i + 1 : i = 0, 1, \ldots, q-1\}$. To complete the proof of the lemma we will show

$$\left\| \frac{1}{N} \sum_{j=0}^{N-1} f \circ \varphi_{p/q}^{j} - c \right\| < \epsilon. \tag{III.3}$$

For this purpose let $[x, s] \in \widetilde{X}$ with $0 \leq s < 1$. Choose $i \in \{0, 1, \ldots, q-1\}$ such that $\{ip/q\} \leq s$ and $|\{ip/q\} - s| < \dfrac{1}{q} < \delta_1$. Then

$$\left|\frac{1}{N}\sum_{j=0}^{N-1} f\circ\varphi_{p/q}^{j}\left([x,s]\right)-c\right| \leq \left|\frac{1}{N}\sum_{j=0}^{N-1} f\circ\varphi_{p/q}^{j}([x,s]) - \frac{1}{N}\sum_{j=0}^{N-1} f\circ\varphi_{p/q}^{j}\left([x,\{ip/q\}]\right)\right|$$

$$+\left|\frac{1}{N}\sum_{j=0}^{N-1}(f\circ\varphi_{p/q}^{j}\left([x,\{ip/q\}]\right)-c)\right|$$

$$=\frac{1}{N}\sum_{j=0}^{N-1}\left|f\left([x,s+jp/q]\right)-f\left([x,\{ip/q\}+jp/q]\right)\right|$$

$$+\left|\frac{1}{N}\sum_{j=0}^{N-1}(T_{i+j}(h^{-[ip/q]}(x))-c)\right|$$

$$<\frac{1}{N}\sum_{j=0}^{N-1}\left|f\left([x,s+jp/q]\right)-f\left([x,\{ip/q\}+jp/q]\right)\right|+\frac{\epsilon}{2}.$$

But since $\left[s+\frac{jp}{q}\right]=\left[\{\frac{ip}{q}\}+\frac{jp}{q}\right]$, we see that

$$\begin{aligned}\max\{\mathrm{d}'(h^{[s+jp/q]}(x),h^{[\{ip/q\}+jp/q]}(x)),\ &|\{s+jp/q\}-\{\{ip/q\}+jp/q\}|\}\\ &=|\{s+jp/q\}-\{\{ip/q\}+jp/q\}|\\ &=s-\{ip/q\}\\ &<\delta_1\end{aligned}$$

and so

$$\begin{aligned}&\mathrm{d}([x,s+jp/q],[x,\{ip/q\}+jp/q])\\ &\quad=\mathrm{d}([h^{[s+jp/q]}(x),\{s+jp/q\}],\ [h^{[\{ip/q\}+jp/q]}(x),\{\{ip/q\}+jp/q\}])\\ &\quad<\delta.\end{aligned}$$

Hence $|f([x,s+jp/q])-f([x,\{ip/q\}+jp/q])|<\dfrac{\epsilon}{2}$. Thus

$$\left|\frac{1}{N}\sum_{j=0}^{N-1}(f\circ\varphi_{p/q}^{j}([x,s])-c)\right|<\epsilon.$$

Since $[x,s] \in \widetilde{X}$ with $0 \leq s < 1$ was chosen arbitrarily, we conclude that (III.3) follows. Thus $\dfrac{p}{q} \in V(f,\epsilon)$ and we are done. $\square$

We are ready to prove the main result of this chapter.

Proposition III.8. *Let (X,h) be a uniquely ergodic dynamical system and let $(\widetilde{X},\varphi)$ be the suspension flow of (X,h). Assume that h^k is uniquely ergodic for some $k > 1$. Then the set*

$$\{\alpha \in [0,1] \colon \varphi_\alpha \textit{ is uniquely ergodic } \}$$

is a dense G_δ set.

Proof. Let $\{f_i\}_{i\in\mathbb{Z}_+}$ be a dense set in $C(\widetilde{X})$. By Lemma III.3, the set

$$\{\alpha \in [0,1] \colon \varphi_\alpha \text{ is uniquely ergodic}\}$$

is equal to

$$\bigcap_{i\geq 1}\bigcap_{j\geq 1} V(f_i,\tfrac{1}{j})$$

where $V(f_i,\frac{1}{j})$ is as in Definition III.2. But, for each $i \geq 1$ and each $j \geq 1$, the set $V(f_i,\frac{1}{j})$ is open by Lemma III.4 and dense by Lemma III.7. The proposition now follows. $\square$

CHAPTER IV

ENTROPY

Let (X, h) be a dynamical system and let $(\widetilde{X}, \varphi)$ be the suspension flow associated to (X, h). In this chapter we will find a formula to compute the (topological) entropy of the time α map φ_α in terms of the entropy of h. The entropy of a homeomorphism h on a compact metric space is either a nonnegative real number or infinity and is denoted by $\mathrm{h_{top}}(h)$ (see [25, Chapter 7]). Entropy is an invariant of topological conjugacy. The following is a definition in Section 5 of Bowen [2].

Definition IV.1. A uniformly continuous flow on a metric space (X, d) is a family of maps $\{\varphi_\alpha : X \to X\}_{\alpha \geq 0}$ with $\varphi_{\alpha+\beta} = \varphi_\alpha \circ \varphi_\beta$ and such that for any $\alpha_0 > 0$ and $\epsilon > 0$ there is a $\delta > 0$ for which $\mathrm{d}(\varphi_\alpha(x), \varphi_\alpha(y)) < \epsilon$ whenever $0 \leq \alpha \leq \alpha_0$ and $\mathrm{d}(x, y) < \delta$.

Lemma IV.2. *Let (X, φ) be a continuous flow where X is a compact metric space. Then the family of maps $\{\varphi_\alpha\}_{\alpha \geq 0}$ is a uniformly continuous flow.*

Proof. Let $\alpha_0 > 0$ and let $\epsilon > 0$. Let d be a metric on X. Using the uniform continuity of the action $X \times [0, \alpha_0] \to X$, choose $\delta > 0$ such that $\mathrm{d}(\varphi_\alpha(x), \varphi_\beta(y)) < \epsilon$ whenever $\max\{\mathrm{d}(x, y), |\alpha - \beta|\} < \delta$ with $\alpha, \beta \in [0, \alpha_0]$. So if $\mathrm{d}(x, y) < \delta$ then for all $0 \leq \alpha \leq \alpha_0$ it follows that $\max\{\mathrm{d}(x, y), |\alpha - \alpha|\} < \delta$ and therefore $\mathrm{d}(\varphi_\alpha(x), \varphi_\alpha(y)) < \epsilon$. $\square$

Lemma IV.3. *Let* (X, φ) *be a continuous flow where* X *is a compact metric space. Let* $\alpha \in \mathbb{R}$*. Then*

$$\mathrm{h}_{\mathrm{top}}(\varphi_\alpha) = |\alpha| \mathrm{h}_{\mathrm{top}}(\varphi_1).$$

Proof. Since $\{\varphi_\alpha\}_{\alpha \geq 0}$ is a uniformly continuous flow by Lemma IV.2, we may use [2, Proposition 21] to obtain for all $\alpha > 0$

$$\mathrm{h}_{\mathrm{top}}(\varphi_\alpha) = \alpha \mathrm{h}_{\mathrm{top}}(\varphi_1).$$

Then

$$\mathrm{h}_{\mathrm{top}}(\varphi_{-\alpha}) = \mathrm{h}_{\mathrm{top}}(\varphi_\alpha^{-1}) = \alpha \mathrm{h}_{\mathrm{top}}(\varphi_1^{-1}) = \alpha \mathrm{h}_{\mathrm{top}}(\varphi_1)$$

where the last equality follows since $\widetilde{X}$ is compact (cf. [25, Theorem 7.3]). Furthermore, since $\varphi_0 = \mathrm{id}$, we have $\mathrm{h}_{\mathrm{top}}(\varphi_0) = 0$. Hence $\mathrm{h}_{\mathrm{top}}(\varphi_\alpha) = |\alpha| \mathrm{h}_{\mathrm{top}}(\varphi_1)$ for all $\alpha \in \mathbb{R}$, as wanted. □

Lemma IV.4. *Let* (X, h) *be a dynamical system and let* $(\widetilde{X}, \varphi)$ *be the suspension flow of* (X, h)*. Then*

$$\mathrm{h}_{\mathrm{top}}(\varphi_1) = \mathrm{h}_{\mathrm{top}}(h).$$

Proof. Consider the canonical quotient map $\pi : X \times \mathbb{R} \to \widetilde{X}$. Set

$$Y = \pi\left(X \times \left[0, \tfrac{1}{2}\right]\right)$$

and

$$Z = \pi\left(X \times \left[\tfrac{1}{2}, 1\right]\right).$$

Then Y and Z are closed φ_1-invariant subsets of $\widetilde{X}$. Furthermore, $\widetilde{X} = Y \cup Z$. Hence, by [13, Proposition 3.1.7(2)],

$$\mathrm{h_{top}}(\varphi_1) = \max\{\mathrm{h_{top}}(\varphi_{1|_Y}), \mathrm{h_{top}}(\varphi_{1|_Z})\}.$$

Now observe that $\pi|_{X\times\left[0,\frac{1}{2}\right]}$ is a homeomorphism from $X \times \left[0, \frac{1}{2}\right]$ to Y. Moreover

$$\varphi_1 \circ \pi([x,t]) = [x, t+1] = [h(x), t] = \pi \circ (h \times \mathrm{id})([x,t]).$$

Hence, $(X \times \left[0, \frac{1}{2}\right], h \times \mathrm{id})$ is conjugate to (Y, φ_1). Thus, using [13, Proposition 3.1.7(4)],

$$\begin{aligned} \mathrm{h_{top}}(\varphi_{1|Y}) &= \mathrm{h_{top}}(\varphi \times \mathrm{id}) \\ &= \mathrm{h_{top}}(h) + \mathrm{h_{top}}(\mathrm{id}) \\ &= \mathrm{h_{top}}(h). \end{aligned}$$

Similarly $\mathrm{h_{top}}(\varphi_{1|Z}) = \mathrm{h_{top}}(h)$. Hence $\mathrm{h_{top}}(\varphi_1) = \mathrm{h_{top}}(h)$. □

We are ready to prove the main result of this chapter.

Theorem IV.5. *Let (X, h) be a dynamical system and let $(\widetilde{X}, \varphi)$ be the suspension flow of (X, h). For every $\alpha \in \mathbb{R}$ it follows that*

$$\mathrm{h_{top}}(\varphi_\alpha) = |\alpha|\mathrm{h_{top}}(h).$$

Proof. This follows immediately from Lemmas IV.3 and IV.4.

□

CHAPTER V

EXAMPLES

In this final chapter we include examples (and counterexamples) about the theory studied in this thesis.

Example 1. Let (X, h) be a dynamical system and let $(\widetilde{X}, \varphi)$ be the suspension flow of (X, h). Chapters 2 and 3 gave sufficient conditions for the existence of minimal and uniquely ergodic time α maps. Is it possible to determine precisely for which α the time α map is minimal and uniquely ergodic? We provide here an example when the answer is affirmative. Let (X, h) be a Denjoy system with irrational rotation number β. We will explain briefly what this means below. We refer to [22] for a detailed account.

Let h be an orientation preserving homeomorphism of the circle S^1 with no periodic points and let β be the irrational rotation number of h. By [22, Corollary 3.2] the homeomorphism h is uniquely ergodic and there exists a continuous surjective map $g\colon S^1 \to S^1$ such that $g \circ h = R_\beta \circ g$, where $R_\beta\colon x \mapsto x + \beta$ is the rigid rotation on $S^1 \cong \mathbb{R}/\mathbb{Z}$. We say that h is a Denjoy homeomorphism of S^1 when the semiconjugation map g just described is not one-to-one. If h is a Denjoy homeomorphism of S^1 with irrational rotation number β then [22, Proposition 3.4] shows that the support X of the unique h invariant probability measure on S^1

is a Cantor set, X is h invariant and the homeomorphism $h\colon X \to X$ is minimal. We then refer to the minimal and uniquely ergodic dynamical system (X, h) as a Denjoy system. Following the notation in [22], suppose that the Cantor set X is obtained from $S^1 \cong \mathbb{R}/\mathbb{Z}$ by removing a countable disjoint union of open intervals $I_1, I_2, \ldots$. To be more specific, let $X = S^1 \setminus \bigcup_{n=1}^{\infty} I_n$, where $\bigcup_{n=1}^{\infty} I_n$ is a countable disjoint union of open intervals and the intervals $I_n = (a_n, b_n)$, for $n \in \mathbb{Z}_+$, are the components of $S^1 \setminus X$. Observe that the semiconjugation map g collapses each of the intervals $I_n = (a, b)$ into a single point. By continuity $g(a_n) = g(b_n)$. Set $T = \{a_n, b_n \colon n \in \mathbb{Z}_+\}$ and $Q(h) = g(T) = \{g(I_n)\colon n \in \mathbb{Z}_+\}$. It follows that the set $Q(h)$ is uniquely determined by h up to a rigid rotation and it is countable and invariant under R_β. Let $1 \leq n(h) \leq \aleph_0$ be the number of disjoint orbits of $Q(h)$ under R_β. Then $Q(h) = \bigcup_{i=1}^{n(h)} Q_i$, where $Q_1, \ldots, Q_{n(h)}$ are the $n(h)$ disjoint R_β invariants orbits of $Q(h)$. So we have

$$Q_i = \{\gamma_i + n\beta \colon n \in \mathbb{Z}\}$$

for $i = 1, \ldots, n(h)$, where $\gamma_i - \gamma_j \notin \{n\beta \colon n \in \mathbb{Z}\}$ when $i \neq j$. As pointed out in Section 3 of [22], we may always assume $\gamma_1 = 0$. Notice now that the set $X \setminus T$ is dense in X and so

$$A = [(X \setminus T) \times \mathbb{R}]/\mathbb{Z}$$

is dense in $\widetilde{X}$.

Proposition V.1. *Let (X, h) be a Denjoy system with irrational rotation number β and let $g : S^1 \to S^1$ be the semiconjugation map satisfying $g \circ h = R_\beta \circ g$, cf. [22].*

Let $(\widetilde{X}, \varphi)$ be the suspension flow of (X, h) and let α be a real number. Identify S^1 with $\mathbb{R}/\mathbb{Z}$. Then the map $F : \widetilde{X} \to S^1 \times S^1$ given by

$$F([x,t]) = (g(x) + \beta t, t)$$

defines an almost one to one extension (cf. Appendix) between the dynamical systems $(\widetilde{X}, \varphi_\alpha)$ and $(S^1 \times S^1, R_{\alpha\beta} \times R_\alpha)$.

Proof. We have

$$\begin{aligned} F([h^n(x), t-n]) &= (gh^n(x) + \beta(t-n), t-n) \\ &= (R^n_\beta g(x) + \beta t - \beta n, t-n) \\ &= (g(x) + \beta n + \beta t - \beta n, t-n) \\ &= (g(x) + \beta t, t-n) \\ &= F([x,t]). \end{aligned}$$

Hence F is well defined. Now, since

$$\begin{aligned} F\varphi_\alpha([x,t]) &= F([x, t+\alpha]) \\ &= (g(x) + \beta t + \alpha\beta, t+\alpha) \\ &= (R_{\alpha\beta} \times R_\alpha)(g(x) + \beta t, t) \\ &= (R_{\alpha\beta} \times R_\alpha)F([x,t]) \end{aligned}$$

and F is clearly onto (because g is onto), it follows that F is an extension map. Let A be the subset $[(X \setminus T) \times \mathbb{R}]/\mathbb{Z}$ of $\widetilde{X}$ with T as defined above. Since A is dense and

$$A = \left\{ [x,s] \in \widetilde{X} \colon F^{-1}F([x,s]) = \{[x,s]\} \right\},$$

we conclude that F is an almost one-to-one extension. $\square$

Proposition V.2. *Let (X,h) be a Denjoy system with irrational rotation number β. Let φ_α be the time α map on the suspension of (X,h). Then the following are equivalent*

1. φ_α *is minimal.*
2. φ_α *is uniquely ergodic.*
3. 1, α *and* $\alpha\beta$ *are linearly independent over* $\mathbb{Q}$.
4. $\alpha \notin \{0\} \cup \left\{ \frac{1}{r_1\beta + r_2} : (r_1, r_2) \in \mathbb{Q}^2 \setminus (0,0) \right\}$.

Proof. Let F be the map as in Proposition V.1 and let $A = [(X \setminus T) \times \mathbb{R}]/\mathbb{Z} \subset \widetilde{X}$ with T as described above. Let λ be the Lebesgue measure on $\mathbb{R}$ and consider the $R_{\alpha\beta} \times R_\alpha$ invariant measure $\lambda \times \lambda$ on $S^1 \times S^1$, where we identify S^1 with $[0,1)$. We claim that $F((T \times \mathbb{R})/\mathbb{Z}) = F(\widetilde{X} \setminus A)$ has measure zero. Indeed, using the notation described before Proposition V.1 we get

$$
\begin{aligned}
(\lambda \times \lambda)(F((T \times \mathbb{R})/\mathbb{Z})) &= \\
&\int_{[0,1)} \lambda(\{x \colon (x,t) \in \{(g(y) + \beta s, s) \colon (y,s) \in T \times [0,1]\}\}) d\lambda(t) \\
&= \int_{[0,1)} \lambda(\{g(y) + \beta t \colon y \in T\}) d\lambda(t) \\
&= \int_{[0,1)} \lambda(Q(h) + \beta t) d\lambda(t) \\
&= \int_{[0,1)} \lambda(Q(h)) d\lambda(t) \\
&= 0.
\end{aligned}
$$

Since $R_\alpha \times R_{\alpha\beta}$ is a group rotation, Proposition 1.4.1 in [13] and Theorem 6.20 in [25] give that $R_\alpha \times R_{\alpha\beta}$ is minimal if and only if $R_\alpha \times R_{\alpha\beta}$ is uniquely ergodic if and only if $1, \alpha, \alpha\beta$ are linearly independent over $\mathbb{Q}$. The equivalence of (1), (2) and (3) is now a consequence of Proposition V.1 and Theorem A.10. To complete the proof of the proposition, we will to show the equivalence of (3) with (4). Set

$$T_1 = \{\alpha \in \mathbb{R} \colon 1, \alpha \text{ and } \alpha\beta \text{ are linearly independent over } \mathbb{Q}\}$$

and

$$T_2 = \{0\} \cup \left\{ \frac{1}{r_1\beta + r_2} \colon (r_1, r_2) \in \mathbb{Q}^2 \setminus (0,0) \right\}.$$

We claim that $T_1 = \mathbb{R} \setminus T_2$. Indeed, suppose that $\alpha \in T_2$. Then if $\alpha = 0$ it is trivial that α and $\alpha\beta$ are not rationally independent. Otherwise, if $\alpha = \dfrac{1}{r_1\beta + r_2}$ for some $(r_1, r_2) \in \mathbb{Q}^2 \setminus (0,0)$, then $r_1\alpha\beta + r_2\alpha - 1 = 0$ and so α and $\alpha\beta$ are not rationally independent. This proves that $T_1 \subset \mathbb{R} \setminus T_2$. For the opposite inclusion, assume that $0 \neq \alpha \notin T_1$; then α and $\alpha\beta$ are not rationally independent. So there exist $(m, n) \in \mathbb{Z}^2 \setminus (0,0)$ and $k \in \mathbb{Z}$ such that $k = m\alpha + n\alpha\beta = \alpha(m + n\beta)$. Since β is irrational, $m + n\beta \neq 0$. Therefore $k \neq 0$ and so $\alpha = \dfrac{k}{n\beta + m} = \dfrac{1}{\frac{n}{k}\beta + \frac{m}{k}}$, proving $T_1 \supset \mathbb{R} \setminus T_2$, as wanted. □

Example 2. Let (X, h) be a dynamical system. We say that (X, h) (or just h) is a group rotation if X is a topological group and if h is rotation by a for some fixed $a \in X$ (i.e. $h(x) = ax$ for all $x \in X$). We say that a group rotation is abelian when X is abelian. In this example we show that the time α map on the suspension of an abelian group rotation is an abelian group rotation. As consequence we will see that the Elliott invariants of the crossed products of such minimal time α maps are actually complete isomorphism invariants.

Lemma V.3. *Let (X, h) be a dynamical system and an abelian group rotation. Let $(\widetilde{X}, \varphi)$ be the suspension flow of (X, h). Let α be a real number. Then the dynamical system $(\widetilde{X}, \varphi_\alpha)$ is an abelian group rotation.*

Proof. Let e be the identity of X. Regarding $\mathbb{R}$ a group under addition, consider the group $X \times \mathbb{R}$ with coordinatewise operation. Since h is a group rotation there is $a \in X$ such that $h(x) = ax$ for all $x \in X$. Let H be the normal subgroup of $X \times \mathbb{R}$ generated by $(a, -1)$. Then $\widetilde{X}$ is equal to $(X \times \mathbb{R})/H$ and so it is an abelian group. The time α map is readily verified to be rotation by $[e, \alpha]$. Hence the dynamical system $(\widetilde{X}, \varphi_\alpha)$ is an abelian group rotation. □

If (X, h) is a dynamical system and a group rotation then X is abelian when either the Haar measure is an ergodic h invariant measure (cf. [25, Theorem 1.9]) or when h is minimal (cf. [25, Theorem 3.4]).

Proposition V.4. *Let (X_1, h_1) be a dynamical system and an abelian group rotation. Let α be a real number and let S_1 be the time α map on the suspension $\widetilde{X}_1$ of (X_1, h_1). We have the following.*

1. *S_1 is minimal if and only if S_1 is uniquely ergodic.*

2. *Suppose that* (X_2, h_2) *is another dynamical system and an abelian group rotation. Assume also that* X_2 *abelian. Let* S_2 *be the time* α *map on the suspension* $\widetilde{X}_2$ *of* (X_2, h_2). *Assume that* S_i *is minimal for* $i = 1, 2$. *Then* $C(\widetilde{X}_1) \rtimes_{S_1} \mathbb{Z}$ *and* $C(\widetilde{X}_2) \rtimes_{S_2} \mathbb{Z}$ *have the same Elliott invariants if and only if they are isomorphic.*

Proof. Since $(\widetilde{X}, S_1)$ is a dynamical system and a group rotation by Lemma V.3, part (1) follows from [25, Theorem 6.20]. Part (2) will follow from [10, Theorem 6] once we verify that $(\widetilde{X}_i, S_i)$ is minimal and equicontinuous for $i = 1, 2$. But minimality is part of the hypothesis and equicontinuity is a consequence of [10, Theorem 2] because $(\widetilde{X}_i, S_i)$ is a minimal dynamical system and a group rotation by Lemma V.3. □

Example 3. Consider the trivial dynamical system (X, h) where X is the one point space. It is obvious that (X, h) is a group rotation. Let α and β be irrational numbers such that the time α map φ_α and the time β map φ_β on the suspension $\widetilde{X}$ of (X, h) are minimal (and so they are also uniquely ergodic by Proposition V.4). By Theorem I.18, the Elliott invariants of $C(\widetilde{X}) \rtimes_{\varphi_\alpha} \mathbb{Z}$ and $C(\widetilde{X}) \rtimes_{\varphi_\beta} \mathbb{Z}$ are the same if and only if $\mathbb{Z} + \alpha\mathbb{Z} = \mathbb{Z} + \beta\mathbb{Z}$ if and only if α has the same image as $\pm\beta$ in $\mathbb{R}/\mathbb{Z}$. Hence using Proposition V.4 we conclude that $C(\widetilde{X}) \rtimes_{\varphi_\alpha} \mathbb{Z}$ and $C(\widetilde{X}) \rtimes_{\varphi_\beta} \mathbb{Z}$ are isomorphic if and only if α has the same image as $\pm\beta$ in $\mathbb{R}/\mathbb{Z}$. Since $C(\widetilde{X}) \rtimes_{\varphi_\alpha} \mathbb{Z}$ is isomorphic to the irrational rotation algebra $A_\alpha = C(S^1) \rtimes_{R_\alpha} \mathbb{Z}$ (likewise with β), we have obtained another way to distinguish irrational rotation algebras, a result obtained originally by Rieffel, Pimsner and Voiculescu [23, 21].

Example 4. Consider the 2-odometer (Y, a) [24, III.5.12]. That is, $T = \prod_{i=0}^{\infty}\{0,1\}$ with the ordinary product topology and a is addition by $\mathbf{1} = (1, 0, \ldots)$ with possibly infinite carry over. Hence (Y, a) is a group rotation. Proposition V.4 then says that the Elliott invariants of crossed products induced by minimal time α maps on the suspension of (Y, a) are complete isomorphism invariants. Our Theorem I.18 gives us these Elliott invariants. We only need to ensure the existence of numbers α making the time α map on the suspension of (Y, a) minimal (and so uniquely ergodic). By Proposition II.12 it suffices to show that a and a^k are minimal for some integer $k > 1$. The following lemma will take care of this.

Lemma V.5. *Let (Y, a) be the 2-odometer. Then a^n is minimal for all odd positive integers n.*

Proof. It is well known that (Y, a) is minimal (and uniquely ergodic) [24, III.5.12]. Let n be an odd integer. We will show that (Y, a^n) is conjugate to (Y, a). We have that a^n is addition by

$$n\mathbf{1} = \underbrace{\mathbf{1} + \mathbf{1} + \ldots + \mathbf{1}}_{n}.$$

Since n is odd, the first component of $n\mathbf{1}$ is equal to 1. Thus the multiplicative inverse $(n\mathbf{1})^{-1}$ of $n\mathbf{1}$ lies in Y (see the proof of [12, Theorem 10.10]). Hence (Y, a^n) is conjugate to (Y, a) via the homeomorphism multiplication by $(n\mathbf{1})^{-1}$. □

We remark that for more general p-odometers with p prime [24, III.5.12], results similar to Lemma V.5 might be obtained.

Example 5. Suppose that (X_1, h_1) and (X_2, h_2) are dynamical systems. Let α be a real number. Let S_i be the time α map on the suspension $\widetilde{X_i}$ of (X_i, h_i) for $i = 1, 2$. If (X_1, h_1) is conjugate to (X_2, h_2) it is easy to see that then $(\widetilde{X_1}, S_1)$ is conjugate to $(\widetilde{X_2}, S_2)$. Suppose now that (X_1, h_1) is (strong) orbit equivalent to (X_2, h_2). Does it follow that $(\widetilde{X_1}, S_1)$ is (strong) orbit equivalent to $(\widetilde{X_2}, S_2)$? We show here an example where the answer is negative.

Suppose that (Y, a) is the 2-odometer (see Example 4). Let $t_1 \neq t_2$ be two real numbers with $0 < t_1, t_2 < \log(2)$. By [11, Theorem 16], there are Toeplitz flows (X_1, h_1) and (X_2, h_2) with entropy equal to t_1 and t_2 respectively and with maximal equicontinuous factor (Y, a). Moreover [11, Theorem 16] asserts that (X_i, h_i) is strong orbit equivalent to (Y, a) for $i = 1, 2$. Denote by π_i the extension map of (X_i, h_i) onto (Y, a) for $i = 1, 2$. As noticed in [11, Remark after Corollary 9], π_i is actually an almost one-to-one extension for $i = 1, 2$ (see Appendix). Observe that the systems (X_i, h_i) and (Y, a) are minimal and so Proposition A.8 says that the three definitions of almost one-to-one extension presented in the Appendix are equivalent. Let $k > 1$ be an odd integer. Since (Y, a^k) is minimal by Lemma V.5 and π_i is also an almost one-to-one extension of (X_i, h_i^k) onto (Y, a^k) for $i = 1, 2$, Theorem A.10 shows that h_i^k is minimal for $i = 1, 2$. Using Proposition II.12 we may choose a real number α such that the time α map S_i on the suspension $\widetilde{X_i}$ of (X_i, h_i) is minimal for $i = 1, 2$. By Theorem IV.5 we obtain

$$\mathrm{h_{top}}(S_1) = |\alpha|\mathrm{h_{top}}(h_1) = |\alpha|t_1 \neq |\alpha|t_2 = |\alpha|\mathrm{h_{top}}(h_2) = \mathrm{h_{top}}(S_2). \qquad \text{(V.1)}$$

But since the suspensions $\widetilde{X_i}$ are connected for $i = 1, 2$, if S_1 were orbit equivalent to S_2, they would be flip conjugate [10, Remark after Theorem 6] and hence they would have the same entropy, contradicting (V.1). Hence (X_1, h_1) and (X_2, h_2) are (strong) orbit equivalent but $(\widetilde{X_1}, S_1)$ and $(\widetilde{X_2}, S_2)$ are not.

Example 6. A well known result about S^1 states that two minimal homeomorphism of S^1 are flip conjugate if and only if their associated crossed products are isomorphic. One may ask whether a similar result follow for connected compact metric 1-dimensional spaces non homeomorphic to S^1. We believe that using techniques emanating from this dissertation we may find an example of two minimal dynamical systems on connected compact metric 1-dimensional spaces which are not strong orbit equivalent yet their associated crossed products have the same Elliott invariants.

Consider the dynamical systems $(\widetilde{X_1}, S_1)$ and $(\widetilde{X_2}, S_2)$ obtained in Example 5. We have that $(\widetilde{X_1}, S_1)$ and $(\widetilde{X_2}, S_2)$ are minimal dynamical systems on connected compact metric 1-dimensional spaces and are not strong orbit equivalent. One could use our Theorem I.18 to compute the Elliott invariants of their associated crossed products if the maps S_1 and S_2 were uniquely ergodic. This Elliott invariants would be the same by [9, Theorem 2.1] and Theorem I.18. To be able to use Proposition III.8, one needs the homeomorphisms $h_1^{k_1}$ and $h_2^{k_2}$ of Example 5 to be uniquely ergodic for some $k_1, k_2 > 0$.

Example 7. Let (X, h) be a dynamical system and suppose that the time α map on the suspension of (X, h) is minimal. Does it follow that the time $\alpha + 1$ map is a minimal? We show here an example where the answer is negative. Choose $\beta \in [0,1]$ such that $\dfrac{-\beta}{1+\beta} \notin \{0\} \cup \left\{ \dfrac{1}{r_1\beta + r_2} : (r_1, r_2) \in \mathbb{Q}^2 \setminus (0,0) \right\}$. For example, take β to be a transcendental number. Suppose that (X,h) is a Denjoy system with irrational rotation number β. (See Example 1.) Put $\alpha = \dfrac{-\beta}{1+\beta}$. Then $\alpha + 1 = \dfrac{1}{1+\beta}$. Thus, by Proposition V.2, φ_α is minimal but $\varphi_{\alpha+1}$ is not.

Example 8. The suspension flow of a dynamical system is a particular case of a more general construction called the suspension under a ceiling function or generalized suspension flow, cf. [24, II.5.14]. Let (Y,a) be a Cantor minimal system and let $F \subset Y$ be clopen. Consider the generalized suspension Y_f of (Y,a) under the ceiling function $f = \chi_{X \setminus F} + \gamma\chi_F$, where $\gamma > 0$. Let φ_α be the time α map on Y_f. We show here that asking $1, \alpha, \gamma$ to be linearly independent over $\mathbb{Q}$ is not sufficient to make φ_α minimal.

Let $\alpha > 0$ and $\beta \in [0,1]$ be irrational numbers such that $1, \alpha, \alpha\beta$ are linearly independent over $\mathbb{Q}$ and $\gamma = \alpha\beta - 1 > 0$. For example, take $\alpha = \sqrt{3}$ and $\beta = \dfrac{1}{\sqrt{2}}$. Let (X,h) be a Denjoy system with irrational rotation number β. (See Example 1.) Consider the dynamical system (Y,a) where $Y = X \times \{0,1\}$ and a is defined by

$$a(x,i) = \begin{cases} (x,1) & \text{if } i = 0 \\ (h(x),0) & \text{if } i = 1 \end{cases}$$

Then (Y,a) is a minimal Cantor system. Let $F = X \times \{0\}$ and let $f = \chi_{Y\setminus F} + \gamma\chi_F$. We have that the time $\dfrac{1}{\beta} = \dfrac{\alpha}{1+\gamma}$ map on the suspension of (X,h) is not minimal (because $1, \dfrac{1}{\beta}, \dfrac{1}{\beta}\beta$ are not linearly independent over $\mathbb{Q}$, cf. Proposition V.2) and is conjugate to the time α map on the generalized suspension Y_f of (Y, a) under the ceiling function f, with conjugation map $[(x,i),t] \mapsto \left[x, \frac{i+t}{\gamma+1}\right]$. We conclude that the time α map on Y_f is not minimal even though $1, \alpha, \gamma$ are linearly independent over $\mathbb{Q}$ (because $\gamma = \alpha\beta - 1$ and $1, \alpha, \alpha\beta$ are linearly independent over $\mathbb{Q}$ by choice). I am thankful to T. Katsura for showing me the idea of the construction used in this example.

Example 9. Let (X, h) be a minimal and uniquely ergodic dynamical system. Let $(\widetilde{X}, \varphi)$ be the suspension flow of (X, h). Let τ be the unique normalized trace of $C(X) \times_h \mathbb{Z}$. Theorem I.18 might suggest a connection between the minimality of φ_α and the triviality of the intersection $\mathbb{Z} \cap \alpha\tau(K_0(C(X) \rtimes_h \mathbb{Z}))$. That is, assuming that φ_α is minimal, does it follow that $\mathbb{Z} \cap \alpha\tau(K_0(C(X) \rtimes_h \mathbb{Z})) = \{0\}$? We show here an example where the answer is negative.

Suppose that $\beta \in [0,1]$ and γ_2 are irrational numbers such that 1, β, γ_2 are linearly independent over $\mathbb{Q}$. For example, take $\beta = \frac{1}{\sqrt{2}}$ and $\gamma_2 = \sqrt{3}$. Let (X, h) be a Denjoy system with irrational rotation number β and $Q(h) = \{n\beta \colon n \in \mathbb{Z}\} \cup \{\gamma_2 + n\beta \colon n \in \mathbb{Z}\}$. (See Example 1.) Then Theorem 5.2 in [22] gives

$$\tau(K_0(C(X) \rtimes_h \mathbb{Z})) = \mathbb{Z} + \mathbb{Z}\beta + \mathbb{Z}\gamma_2.$$

Put $\alpha = \frac{1}{\gamma_2}$. Using Proposition V.2 we obtain that φ_α is minimal (because $1, \alpha, \alpha\beta$ are linearly independent over $\mathbb{Q}$ since $1, \beta, \gamma_2 = \frac{1}{\alpha}$ are linearly independent over $\mathbb{Q}$ by choice). But

$$\mathbb{Z} \cap \alpha\tau K_0((C(X) \rtimes_h \mathbb{Z})) = \mathbb{Z}.$$

APPENDIX A

ALMOST ONE-TO-ONE EXTENSIONS

Let (X, φ) and (Y, ψ) be dynamical systems. We say that (X, φ) is an extension of (Y, ψ) if there is a continuous map π of X onto Y such that $\pi \circ \varphi = \psi \circ \pi$. We call π an extension map. If the extension map π were also one-to-one, then π would be a homeomorphism and so (X, φ) would be conjugate to (Y, ψ). It is clear that if two dynamical systems are conjugate, then one is minimal (uniquely ergodic) if and only if the other one is also minimal (uniquely ergodic). If π is not one-to-one, one may hope for π to be "almost" one-to-one. Once we have a satisfactory definition of what we mean by almost one-to-one, we may proceed to ask, as in the case when π was one-to-one, whether minimality (unique ergodicity) of one of the dynamical systems implies minimality (unique ergodicity) of the other. In this appendix we propose a definition for almost one-to-one extensions, compare it with those (seemingly different) definitions of almost one-to-one extensions found in the literature and show that all of them coincide when the dynamical systems involved are minimal. We conclude with a result giving conditions for when an extension of dynamical systems preserves minimality (unique ergodicity). The results in this appendix might well be known to the specialists; however we decided to include them anyway as we could not find them stated in the literature.

The following is our proposed definition of an almost one-to-one extension.

Definition A.1. We say that an extension $\pi : X \to Y$ is almost one-to-one if the set $A = \{x \in X : \pi^{-1}\pi(x) = \{x\}\}$ is dense.

Other definitions found in the literature are the following.

Definition A.2 ([24, 4.6.1(2)]). An extension $\pi : X \to Y$ is almost one-to-one if there is a point x in X with dense orbit such that $\pi^{-1}\pi(x) = \{x\}$.

Definition A.3 ([4, Page 152]). We say that an extension $\pi : X \to Y$ is almost one-to-one if the set $B = \{y \in Y : |\pi^{-1}(y)| = 1\}$ contains a dense G_δ set. The symbol $|E|$ means the cardinality of the set E.

Lemma A.4. *Let $\pi : X \to Y$ be an extension map. Let A be as in Definition A.1. Then $\pi|_A$ is one-to-one.*

Proof. If $a_1, a_2 \in A$ and $\pi(a_1) = \pi(a_2)$, then $\{a_1\} = \pi^{-1}\pi(a_1) = \pi^{-1}\pi(a_2) = \{a_2\}$ and thus $a_1 = a_2$. □

Lemma A.5. *Let $\pi : X \to Y$ be an extension map. Let A be as in Definition A.1 and let B be as in Definition A.3. Then $A = \pi^{-1}(B)$ and $\pi(A) = B$.*

Proof. If $a \in A$ then $\pi^{-1}\pi(a) = \{a\}$, that is, $|\pi^{-1}(\pi(a))| = 1$, so $\pi(a) \in B$. Hence $a \in \pi^{-1}(B)$. Conversely, if $a \in \pi^{-1}(B)$ then $\pi(a) \in B$, that is, $|\pi^{-1}(\pi(a))| = 1$. It follows that $\pi^{-1}\pi(a) = \{a\}$. This proves that $A = \pi^{-1}(B)$. The other equality is now immediate. □

Lemma A.6. *Let* (X, φ) *be an extension of* (Y, ψ) *with extension map* $\pi \colon X \to Y$*. The sets* A *and* B *in Definitions A.1 and A.3 are invariant sets in* X *and* Y*, respectively.*

Proof. It suffices to show the next two statements.

(i) $\psi(B) \subset B$ and

(ii) $\psi^{-1}(B) \subset B$.

Indeed, this would give us that $\psi(B) = B = \psi^{-1}(B)$ and so

$$\varphi^{-1}(A) = \varphi^{-1}\pi^{-1}(B) = \pi^{-1}\psi^{-1}(B) = \pi^{-1}(B) = A$$

as wanted.

To prove (i), let $b \in B$. Then $|\varphi^{-1}\pi^{-1}\psi(b)| = |\pi^{-1}\psi^{-1}\psi(b)| = |\pi^{-1}(b)| = 1$. From this we obtain that $|\pi^{-1}\psi(b)| = 1$, that is, $\psi(b) \in B$, as was to be proved.

The proof of (ii) is analogous. $\square$

Proposition A.7. *Let* (X, φ) *be an extension of* (Y, ψ) *with extension map* $\pi : X \to Y$*. We have the following.*

(i) If π *is an almost one-to-one extension in the sense of Definition A.2 then* π *is an almost one-to-one extension in the sense of Definition A.1. The converse is false.*

(ii) If π is an almost-one-to-one extension in the sense of Definition A.1 then π is an almost one-to-one extension in the sense of Definition A.3. The converse is false.

Proof. To prove (i), assume π is an almost one-to-one extension in the sense of Definition A.2. Then there is x in X with dense orbit and $\pi^{-1}\pi(x) = \{x\}$. Let A be as in A.1. Then x belongs to A. By Lemma A.6 the orbit of x is a subset of A. Hence A is dense. Thus π is one-to-one in the sense of Definition A.1. To show that the converse is false, consider the following example. Let $X = Y$ consist of at least two points and put $\varphi = \psi = \pi = \mathrm{Id}_X$. Then π is an extension map which is almost one-to-one in the sense of Definition A.1 (because $A = X$) but is not one-to-one in the sense of Definition A.2 (no point in X has dense orbit).

For (ii), assume that π is an almost one-to-one extension in the sense of Definition A.1. Let A be as in A.1 and let B be as in A.3. By Lemma A.5 we have $B = \pi(A)$. Hence B is dense since A is dense. Moreover, B is a G_δ set from a remark in [24, VI.6.4(1)]. Thus π is an almost one-to-one extension in the sense of Definition A.3. To show that the converse is false, consider the following example. Let $X = [0, 1] \cup \{2\}$ and let $Y = [0, 1]$. Put $\varphi = \mathrm{Id}_X$, $\psi = \mathrm{Id}_Y$ and define $\pi\colon X \to Y$ to be the identity on $[0, 1]$ and $\pi(2) = 1$. Then π is an extension map which is almost one-to-one in the sense of Definition A.3 (because $B = Y$) but is not almost one-to-one in the sense of Definition A.1 (because $A = [0, 1]$ is not dense in $X = [0, 1] \cup \{2\}$). □

Proposition A.8. *If $\pi\colon X \to Y$ is an extension of minimal systems then the Defi-*

nitions A.1, A.2 and A.3 are equivalent to each other.

Proof. Suppose that $\pi\colon X \to Y$ is an extension of minimal systems. By [1, Theorem 1.15], π is semi-open, that is, if $U \neq \emptyset$ is open in X then $\pi(U)$ has non-empty interior. Thus A.3 implies A.1.

Let A be as in A.1. If A is dense, it is nonempty. Then there exists $x \in X$ such that $\pi^{-1}\pi(x) = \{x\}$. The orbit of x is dense since φ is minimal. Thus A.1 implies A.2. □

Lemma A.9. *Suppose that $\varphi\colon X \to X$ is a minimal and uniquely ergodic homeomorphism with μ the unique invariant probability measure. Let A be a subset of X. If $\mu(A) = 1$ then A is dense.*

Proof. If $U \subset X$ is open then $\mu(A \cap U) = \mu(A) + \mu(U) - \mu(A \cup U) = \mu(U)$. But $\mu(U) > 0$ by [25, Theorem 6.17]. Hence $A \cap U \neq \emptyset$. □

Theorem A.10. *Let (X, φ) and (Y, ψ) be two dynamical systems. Suppose that (X, φ) is an extension of (Y, ψ) with extension map $\pi\colon X \to Y$. Let ν be an invariant probability measure on Y and consider the set $A = \{x \in X\colon \pi^{-1}\pi(x) = \{x\}\}$.*

(a) If A is dense (i.e. π is an almost one-to-one extension in the sense of A.1) then φ is minimal if and only if ψ is minimal.

(b) If $\nu(\pi(A)) = 1$ then φ is uniquely ergodic if and only if ψ is uniquely ergodic.

Proof. If M is a nontrivial closed ψ-invariant subset of Y then $\pi^{-1}(M)$ is a nontrivial closed φ-invariant subset of X. This shows that if (X, φ) is an extension of (Y, ψ) and

φ is minimal, then ψ is minimal. For the converse, we use the hypothesis that the extension is almost one-to-one. Suppose that ψ is minimal. Let M be a nonempty closed φ invariant subset of X. Then M is compact and so $\pi(M)$ is compact. Hence $\pi(M)$ is a nonempty closed ψ invariant subset of Y. Thus $\pi(M) = Y$. Since A is dense, the open set $X \setminus M$ must be empty; otherwise, there is an element x in $A \cap (X \setminus M)$ so $\pi(x) \in Y = \pi(M)$. This means that $\pi(x) = \pi(y)$ for some $y \in M$ and hence $\{x\} = \pi^{-1}\pi(x) = \pi^{-1}\pi(y)$. Thus $x = y \in M$, a contradiction.

Let us prove now (b). If φ is uniquely ergodic then ψ is uniquely ergodic by [24, Corollary IV.1.8]. For the converse, we use the hypothesis that $\nu(\pi(A)) = 1$. Assume ν is the only invariant probability measure on Y. Then there is an invariant probability measure on X, call it μ, satisfying $\nu = \mu \circ \pi^{-1}$ (cf. [24, Corollary 4.1.9]). Let μ_0 be another invariant probability measure on X. Then $\mu_0 \circ \pi^{-1}$ must be equal to ν. Let $E \subset X$ be an arbitrary measurable set. Since π is one-to-one when restricted to A by Lemma A.4, we have $E \cap A = \pi^{-1}\pi(E \cap A)$. Therefore $\mu_0(E \cap A) = \mu_0\pi^{-1}\pi(E \cap A) = \nu\pi(E \cap A)$. Furthermore, using Lemma A.5 and the hypothesis,

$$
\begin{aligned}
\mu_0(A) &= \mu_0(\pi^{-1}(B)) \\
&= \nu(B) \\
&= \nu(\pi(A)) \\
&= 1.
\end{aligned}
$$

This gives $\mu_0(X \setminus A) = 0$ and so $\mu_0(E \cap (X \setminus A)) = 0$. Furthermore, $\mu(A) = 1$ implies $\mu(E \cap (X \setminus A)) = 0$. Hence

$$\begin{aligned}
\mu_0(E) &= \mu_0((E \cap A) \cup (E \cap (X \setminus A))) \\
&= \mu_0(E \cap A) + \mu_0(E \cap (X \setminus A)) \\
&= \nu(\pi(E \cap A)) \\
&= \mu\pi^{-1}\pi(E \cap A) \\
&= \mu(E \cap A) \\
&= \mu(E \cap A) + \mu(E \cap (X \setminus A)) \\
&= \mu(E).
\end{aligned}$$

Thus $\mu = \mu_0$. Hence φ is uniquely ergodic. □

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