Please do any 10 problems out of the following 20.

1. Define when a pair of topological spaces \((X,Y)\) is a Borsuk pair. Prove that a \(CW\)-pair \((X,Y)\) is a Borsuk pair (in the case when \(X, Y\) are finite complexes).

2. Define covering space. Prove that any map \(f : \mathbb{R}P^2 \rightarrow S^1\) is homotopic to a constant map.

3. Let \(p : E \rightarrow B\) be a Serre fiber bundle, where \(B\) is a path connected space. Prove that for any two points \(x_0, x_1 \in B\) the fibers \(F_0 = p^{-1}(x_0)\) and \(F_1 = p^{-1}(x_1)\) are weak homotopy equivalent.

4. State the Lefschetz Fixed Point Theorem. Let

\[
f : \mathbb{C}P^{4k} \times \mathbb{R}P^2 \times \mathbb{R}P^{2n} \rightarrow \mathbb{C}P^{4k} \times \mathbb{R}P^2 \times \mathbb{R}P^{2n}
\]

be a map. Prove that \(f\) always has a fixed point.

5. Define the Whitehead map \(w : S^n \vee S^k \rightarrow S^n \vee S^k\). Prove that the element \([w] \in \pi_{n+k-1}(S^n \vee S^k)\) is in the kernel of the suspension homomorphism

\[
\Sigma : \pi_{n+k-1}(S^n \times S^k) \rightarrow \pi_{n+k}(\Sigma(S^n \times S^k)).
\]

6. Let \(A : S^n \rightarrow S^n\) be the antipodal map, \(A : x \mapsto -x\), and \(\iota_n \in \pi_n(S^n)\) be the generator represented by the identity map \(S^n \rightarrow S^n\). Prove that the homotopy class \([A] \in \pi_n(S^n)\) is equal to

\[
[A] = \begin{cases} 
\iota_n, & \text{if } n \text{ is odd;} \\
-\iota_n, & \text{if } n \text{ is even.}
\end{cases}
\]

7. Compute the ring structure \(H^*(\mathbb{R}P^{2n+1}; \mathbb{Z}/8)\).

8. Let \(X \subseteq S^n\) be homeomorphic to \(S^p \vee S^q\), \(1 \leq p, q \leq n-1\). Compute the homology groups \(\tilde{H}_q(S^n \setminus X)\).

9. Let \(X\) be a finite simply-connected \(CW\)-complex with \(\tilde{H}_n(X) = 0\) for all \(n\). Prove that \(X\) is contractible.

10. Let \(X\) be a \(CW\)-complex. Prove that the group \(H^1(X; \mathbb{Z})\) is a free abelian group.
Name: 

ID#: 

There are 10 questions in this exam.

**Problem 1.** The graph of the function

\[ f : (-\pi, \pi) \times (-\pi, \pi) \to \mathbb{R}^1 \rightarrow (x, y) \mapsto e^{x+y} \]

defines a smooth surface \( \Sigma^2 \) in Euclidean space \( \mathbb{E}^3 \).

a) Find an expression for the tangent plane to \( \Sigma^2 \) at the point \( \begin{pmatrix} 1 \\ 1 \\ e^2 \end{pmatrix} \).

b) Calculate the components of the first fundamental form of \( \Sigma^2 \) with the coordinate basis \( \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\} \).

c) Write down an integral expression for the area of the surface \( \Sigma^2 \).

d) Let

\[ \beta : [0, 1] \to \mathbb{R}^2 \quad t \mapsto \begin{pmatrix} t \\ t^2 \end{pmatrix} \]

be a path in \( \mathbb{R}^2 \). Write down an explicit integral expression for the length of the path \( \alpha(t) := \sigma \circ \beta(t) \), where \( \sigma(x, y) = \begin{pmatrix} x \\ y \\ e^{x+y} \end{pmatrix} \) is the map whose image is \( \Sigma^2 \).

**Problem 2.** a) Given a smooth surface \( \Sigma^2 \) in Euclidean space \( \mathbb{E}^3 \), state the definition of the second fundamental form \( K \).

b) Show that \( K \) is an \( \begin{pmatrix} 0 \\ 2 \end{pmatrix} \) tensor on \( \Sigma^2 \).

c) Show that \( K \) is symmetric, in the sense that \( K(v, w) = K(w, v) \) for \( v, w \in T_p \Sigma \).
**Problem 3.** Let $M$ be a smooth manifold and let $\Sigma$ be a subset of $M$.

a) Carefully state the conditions necessary for $\Sigma$ to be an embedded submanifold.

b) Let $M = GL(n, \mathbb{R})$. Show that $SL(n, \mathbb{R}) = \{ A \in GL(n, \mathbb{R}) \mid \det A = 1 \}$ is an embedded submanifold of $GL(n, \mathbb{R})$.

c) Show that $O(n, \mathbb{R}) = \{ A \in GL(n, \mathbb{R}) \mid A^T A = Id \}$ is an embedded submanifold of $GL(n, \mathbb{R})$.

Note: For

\[
\det : GL(n, \mathbb{R}) \rightarrow \mathbb{R} \\
A \mapsto \text{determinant of } A,
\]

\[
D(\det(A))B = \det A \cdot \text{tr}[A^{-1}B].
\]

For

\[
S : GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R}) \\
A \mapsto A^T A,
\]

\[
D(S(A))B = B^T A + A^T B.
\]

**Problem 4.** Let $g = dx^2 + dy^2 + dz^2 + dw^2$ be a Riemannian metric on $\mathbb{R}^4$ and let $\Omega = dx \wedge dz + dy \wedge dw$ be a symplectic 2-form on $\mathbb{R}^4$. Let

\[
i : \mathbb{R}^3 \rightarrow \mathbb{R}^4 \quad (x, y, z) \mapsto (x, y, z, x^2 + y^2 + z^2)
\]

be an embedded submanifold of $\mathbb{R}^4$.

a) Find the pullback of $g$ to $i(\mathbb{R}^3)$. Is it a Riemannian metric? Explain.

b) Find the pullback of $\Omega$ to $i(\mathbb{R}^3)$. Is it a symplectic 2-form? Explain.

c) Write an integral expression for the volume of the hypersurface $i(\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\})$.

d) Let $H : \mathbb{R}^4 \rightarrow \mathbb{R}$ and let $\frac{d}{dt} \gamma = \Omega^{-1}(dH, \cdot)$, $\gamma(0) = p \in \mathbb{R}^4$ determine $\gamma : I \rightarrow \mathbb{R}^4$. Show that $H$ is conserved along $\gamma$.

**Problem 5.** Recall the definition of the Lie derivative of a vector field $W$ along a vector field $V$

\[
L_V W(p) = \lim_{h \to 0} \frac{(\theta_{-h})_* W_{\theta_h(p)} - W_p}{h}
\]
for $\theta_h$ the flow of $V$.

a) For the case $V(p) \neq 0$, show that $(\mathcal{L}_V W)(p) = [V, W](p)$.

b) Show (using part a) that in terms of a local coordinate chart

\[ \mathcal{L}_V W = V^a \frac{\partial}{\partial x^a} W^b - W^a \frac{\partial}{\partial x^a} V^b. \]

c) Verify that for 1-form $\alpha$, $\mathcal{L}_V \alpha = i_V d\alpha + d(i_V \alpha)$.

Problem 6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, and let

\[ g = e^{2f(x, y)} \left( dx^2 + dy^2 \right) \]

be a Riemannian metric on $\mathbb{R}^2$. Compute the scalar curvature for the Levi-Civita (Riemannian) connection corresponding to $g$.

Problem 7. Let $M$ be a smooth, oriented, compact $n$-dimensional manifold with boundary.

a) Given a careful statement of Stokes' Theorem for a smooth $(n - 1)$-form $\omega$ on $M$.

b) Let $N := \mathbb{R}^3 \setminus \{0\}$. Define the following 2-form on $N$:

\[ \alpha = \frac{x dy \wedge dz - y dx \wedge dz + z dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}. \]

b1) Show that $d\alpha = 0$.

b2) Show that $\int_{x^2+y^2+z^2=r^2} \alpha$ is independent of the constant $r$.

b3) Knowing that $\int_{x^2+y^2+z^2=1} \alpha = 4\pi$, compute $\int_{(x-1)^2+(y-2)^2+(z-3)^2=1} \alpha$ and $\int_{(x-1)^2+(y-2)^2+(z-3)^2=25} \alpha$.

b4) Show that there does not exist a 1-form $\beta$ on $N$ for which $d\beta = \alpha$.

Problem 8. Let $\nabla$ be a connection on a Riemannian manifold $(M, g)$ which is metric compatible and has nonzero torsion $Q$.

Find an expression for the Christoffel symbols

\[ \Gamma^a_{bc} = dx^a \left( \nabla_{\frac{\partial}{\partial x^b}} \frac{\partial}{\partial x^c} \right). \]
of this connection in terms of the component of the metric and their derivatives, and in terms of the components for the torsion $Q^a_{bc}$. (carefully show how you obtain this expression)

**Problem 9.** A homogeneous space consists of $(M, G, \phi)$ where $M$ is a smooth manifold, $G$ is a Lie group, and $\phi$ is a transitive action on $M$.

a) Show that

a1) $(S^{n-1}, SO(n), \phi = \text{rotation action})$ is a homogeneous space.

a2) $(\mathbb{R}^{2+}, SL(2, \mathbb{R}), \hat{\phi})$ is a homogeneous space, for

$$\mathbb{R}^{2+} = \{(x, y) | y > 0\}$$

$$SL(2, \mathbb{R}) = \{A \in \mathbb{R}^{2 \times 2} | \det A = 1\}$$

$$\hat{\phi} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (x, y) \right) = \left( \begin{pmatrix} a(x + iy) + b \\ c(x + iy) + d \end{pmatrix}, \text{Re} \end{pmatrix} \left( \begin{pmatrix} a(x + iy) + b \\ c(x + iy) + d \end{pmatrix}, \text{Im} \end{pmatrix} \right).$$

b) Recall the theorem which states that if $(M, G, \phi)$ is a homogeneous space, then for any $p \in M$, there is a diffeomorphism from $G/G_p$ to $M$, where $G_p$ is the isotropy group for $p$. Use this to argue that $SO(3)/SO(2)$ is diffeomorphic to $S^2$.

**Problem 10.** State the Poincare lemma for the deRham cohomology, and prove it (in your proof, you may assume the homotopy equivalence theorem for deRham cohomology).