QUALIFYING EXAM, Fall 2001

Algebraic Topology and Differential Geometry

NAME ___________________________ (PRINT LAST AND THE FIRST NAME)

STUDENT NUMBER ___________________ SIGNATURE ___________________

Please do any 10 problems out of the following 20.

1. Consider the Lie group $SO(3)$. Compute the groups $\pi_i(SO(3))$ for $i = 1, 2, 3$ and the groups $H^q(SO(3); \mathbb{Z}/2)$ for all $q \geq 0$. If you use any homeomorphisms, prove them.

2. Consider the pair $(D^3, S^2)$. Let $K$ be a $CW$-complex, and $E = \mathcal{C}(D^3, K)$, $B = \mathcal{C}(S^2, K)$ (the spaces of continuous maps). The map $p : E \to B$ is defined as $p(f : D^3 \to K) = (f|_{S^2} : S^2 \to K)$. Prove explicitly (i.e. without using general theorem) that the map $p : E \to B$ is a Serre fiber bundle.

3. Define the Hopf invariant $h(\lambda)$ for $\lambda \in \pi_{4q-1}(S^{2q})$. Let $\eta \in \pi_3(S^2)$ be given by the Hopf map $S^3 \to S^2$. Compute the Hopf invariant $h(\eta)$.

4. Let $\alpha \in \pi_n(X)$, $\beta \in \pi_k(X)$, and $\Sigma : \pi_{n+k-1}(X) \to \pi_{n+k}(\Sigma X)$ be the suspension homomorphism. Prove that $[\alpha, \beta] \in \text{Ker } \Sigma$.

5. State the Lefschetz Fixed Point Theorem. Let $f : \mathbb{RP}^{2002} \times \mathbb{RP}^{2000} \to \mathbb{RP}^{2002} \times \mathbb{RP}^{2000}$ be a map. Prove that $f$ always has a fixed point.

6. Give definition of weak homotopy equivalence. Prove that $\mathbb{RP}^n \times S^{n+1}$ is not homotopy equivalent to $\mathbb{RP}^{n+1} \times S^n$.

7. State and prove the Jordan-Brouwer Theorem. (If you would like to use some preliminary results, please state them clearly.)

8. Define a cup-product in cohomology. Let $X = S^n \times S^k$. What is a ring structure of $H^*(X; \mathbb{Z})$? Explain.

9. Let $(X, A)$ be a $CW$-pair. Prove that the group $H^1(X, A; \mathbb{Z})$ is a free abelian group.

10. Define a degree $\deg f$ of a map $f : S^n \to S^n$. Compute $\deg A$ of the antipodal map $A : x \mapsto -x$. 
11. The definitions of the curvature and torsion of a generic path \( \gamma(t) \) in Euclidean space \( \mathbb{R}^3 \) are based on a result that the Frenet frame \( \{e_1, e_2, e_3\} \) satisfies the following relation:

\[
\frac{d}{dt} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & \mu & 0 \\ -\mu & 0 & \nu \\ 0 & -\nu & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}
\] (1)

for a pair of functions \( \mu(t) \) and \( \nu(t) \), where \( \{e_1, e_2, e_3\} \) are the unit tangent, normal, and binormal vectors.

1. Give a careful definition of the Frenet frame of such a curve in \( \mathbb{R}^3 \) and derive equation (1).
2. Specify a generic condition for paths in \( \mathbb{R}^4 \) sufficient for a Frenet frame \( \{e_1, e_2, e_3, e_4\} \) to be defined for such a path.
3. Give the equation comparable to equation (1) above for generic paths in \( \mathbb{R}^4 \)

12. The second fundamental form of a surface \( S \) in \( \mathbb{R}^3 \) is defined as

\[ k(u, v) = -\langle D_u e_\perp, v \rangle \]

where \( e_\perp \) is a surface orthogonal vector field, \( u \) and \( v \) are vectors tangent to the surface, \( D_u \) is the directional derivative along \( u \), and \( \langle \cdot, \cdot \rangle \) is the Euclidean inner product.

1. Show that \( k(v, w) = k(w, v) \).
2. Which of the following surfaces has vanishing second fundamental form? You need not justify your answer and in particular, no calculation is required:
   
   (a) The surface \( S_1 := \{(x, y, z) : x^2 + y^2 + z^2 = 1\} \).
   (b) The surface \( S_2 := \{(x, y, z) : 4x^2 + y^2 = 1\} \).
   (c) The surface \( S_3 := \{(x, y, z) : x^2 + y^2 = z^2, z > 1\} \).

13. Let \( (M, g) \) be a Riemannian manifold, let \( N \) be a manifold, and let \( \psi : N \to M \) be a smooth map. Recall that the “pull-back” \( \psi^* g \) is defined by setting \( \psi^* g(u, v) := g(\psi_* u, \psi_* v) \) for tangent vectors \( u, v \) on \( N \).

1. Prove or disprove the following assertion: “If \( \psi \) is a smooth immersion, then \( \psi^* g \) is a Riemannian metric on \( N \)”.
2. Prove or disprove the following assertion: “If \( \psi \) is a smooth submersion, then \( \psi^* g \) is a Riemannian metric on \( N \)”.

14. (a) Let \( f_i : \mathbb{R}^n \to \mathbb{R}, \ 1 \leq i \leq k \) be functions and let

\[ \Sigma := \{p \in \mathbb{R}^n : f_i(p) = 0, 1 \leq i \leq k\} \]

State a theorem giving precise conditions under which \( \Sigma \) is a closed, embedded submanifold of \( \mathbb{R}^n \) of dimension \( m \).

(b) Let \( \Sigma := \{(w, x, y, z) \in \mathbb{R}^4 : w^2 + x^2 + y^2 + z^2 = 1 \text{ and } x^2 + y^2 + z^2 = w^2\} \). Show, using your theorem above, that \( \Sigma \) is a closed, embedded, two-dimensional submanifold of \( \mathbb{R}^4 \).
15. Let \((M, g)\) be a Riemannian manifold.

1. The Levi-Civita connection is metric compatible and torsion free.
   
   (a) Give a careful definition of what it means for a connection to be metric compatible.
   
   (b) Give a careful definition of what it means for a connection to be torsion free.

2. Let \(\{\partial_a\}\) be a local coordinate basis and let \(\Gamma_{ac}^e\) be the Christoffel symbols of the Levi-Civita connection relative to this basis, i.e.

   \[
   \nabla_{\partial_a} \partial_c = \sum_e \Gamma_{ac}^e \partial_e.
   \]

   Show that

   \[
   \Gamma_{ac}^e = \frac{1}{2} \sum_b g^{eb} (\partial_a g_{bc} + \partial_c g_{ab} - \partial_b g_{ac}).
   \]

3. Show that there is a unique metric compatible torsion free connection on \((M, g)\).

16. Let \(G\) be a Lie group, and for \(g \in G\), define the map \(L_g : G \to G\) by \(L_g(h) = gh\) for \(h \in G\). A 1-form \(\omega\) on \(G\) is left-invariant if \(L_g^*\omega = \omega\) for all \(g \in G\).

   Let \(G = \mathbb{R}^+ \times \mathbb{R} = \{(x, y) : x > 0\}\) with group multiplication given by

   \[(x, y) \cdot (\hat{x}, \hat{y}) = (x\hat{x}, x\hat{y} + y).\]

   Find explicitly all left invariant 1-forms on \(G\), and show that your answer is correct.

17. Let \(M\) be a smooth oriented compact manifold with boundary.

   1. Give a careful statement of Stokes’ theorem for a smooth differential form \(\omega\) on \(M\).

   2. Let \(N := \mathbb{R}^3 - \{0\}\). Define the following 2-form on \(N\):

      \[
      \omega_2 = \frac{xydz - ydx + zdx + dy}{(x^2 + y^2 + z^2)^{3/2}}.
      \]

      (a) Show that \(d\omega_2 = 0\).

      (b) Show that there is no 1 form \(\omega_1\) so \(d\omega_1 = \omega_2\).

      (c) Show that \(\int_{x^2+y^2+z^2=r^2} \omega_2\) is independent of \(r > 0\).

      (d) Given that \(\int_{x^2+y^2+z^2=1} \omega_2 = 4\pi\), compute

      \[
      \int_{(x-1)^2 + (y-2)^2 + (z-3)^2 = 1} \omega_2 \quad \text{and} \quad \int_{(x-1)^2 + (y-2)^2 + (z-3)^2 = 25} \omega_2.
      \]
18. Let \( f(u,v) := (\cos(u)(\cos(v) + 2), \sin(u)(\cos(v) + 2), \sin(v) + 3) \) for \( u, v \in [0, 2\pi] \) parametrize a toroidal surface \( T \) of revolution in \( \mathbb{R}^3 \).

1. Let \( \gamma_u(t) := f(u,t) \) define a curve in \( T \). For what values of \( u \) is this curve a geodesic?
2. Let \( \tau_v(t) := f(t,v) \) define a curve in \( T \). For what values of \( v \) is this curve a geodesic?

19. Let \( \nabla \) be a connection on the vector bundle \( \{B, M, \pi, F\} \). Give the formula for the associated curvature operator

\[
R : TM \times TM \times B \to B
\]

where \( TM \) and \( B \) denote the spaces of sections of the vector bundles \( TM \) and \( B \) respectively. Show that \( R \) is a tensor.

20. Hamilton’s equations of motion for a one-dimensional oscillator take the form

\[ i_v \omega = -dH \text{ where } H : T^*\mathbb{R}^1 \to \mathbb{R}^1 \text{ is } H(x, p) = \frac{p^2}{2} + \frac{k}{2}x^2, \omega = dx \wedge dp, \text{ and } \]

\[ v = \frac{dx}{dt} \frac{\partial}{\partial x} + \frac{dp}{dt} \frac{\partial}{\partial p} \in T(T^*\mathbb{R}^1). \]

1. Derive the explicit ODE system \( \frac{d}{dt} x = \cdots, \frac{d}{dt} p = \cdots \) for this Hamiltonian.
2. Show that \( H(x(t), p(t)) = \frac{p^2(t)}{2} + \frac{k}{2}x^2(t) \) is a constant along any solution.
SOLUTIONS of the problems 1–10

1. Consider the Lie group \( SO(3) \). Compute the groups \( \pi_i(SO(3)) \) for \( i = 1, 2, 3 \) and the groups \( H^q(SO(3); \mathbb{Z}/2) \) for all \( q \geq 0 \).

**Solution.** The space \( SO(3) \) is homeomorphic to \( \mathbb{RP}^3 \). Indeed, for any transformation \( A \in SO(3) \), there exists an orthonormal basis \( e_1, e_2, e_3 \), so that the transformation \( A \) is given by the matrix

\[
A = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{pmatrix}
\]

where \( \pi \leq \theta \leq \pi \). Then we identify \( \mathbb{RP}^3 \) with the disk \( D^3 \) of radius \( \pi \) with the antipodal points identified: \( x \cong -x, \ |x| = \pi \).

The map \( f : SO(3) \to \mathbb{RP}^3 \) is given by \( \theta e_3 \). It is easy to check that \( f \) is a homeomorphism. Thus it is enough to compute the homotopy and homology groups for \( \mathbb{RP}^3 \). Recall that there is a covering map \( p : S^3 \to \mathbb{RP}^3 \). Thus there is an isomorphism \( \pi_q S^3 \cong \pi_q \mathbb{RP}^3 \) for \( q \geq 2 \). We obtain \( \pi_2 \mathbb{RP}^3 = 0 \) and \( \pi_3 \mathbb{RP}^3 = \mathbb{Z} \). The covering map \( p : S^3 \to \mathbb{RP}^3 \) is two-fold, hence the the fundamental group of \( \mathbb{RP}^3 \) is \( \mathbb{Z}/2 \). Different argument: The natural embedding \( \mathbb{RP}^2 \to \mathbb{RP}^3 \) is an embedding of the second skeleton induces of \( \mathbb{RP}^3 \). Thus \( \pi_1 \mathbb{RP}^3 \cong \pi_1 \mathbb{RP}^2 = \mathbb{Z}/2 \).

To compute the homology groups, we use the cellular chain complex. The space \( \mathbb{RP}^3 \) has the cell-decomposition: \( e_0, e_1, e_2, e_3 \). The complex \( \mathcal{E}(\mathbb{RP}^3) \otimes \mathbb{Z}/2 \) is

\[
0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0
\]

Thus \( H_q(\mathbb{RP}^3; \mathbb{Z}/2) = \mathbb{Z}/2 \), for \( q = 0, 1, 2, 3 \), and \( H_q(\mathbb{RP}^3; \mathbb{Z}/2) = 0 \) for \( q > 3 \).

2. Consider the pair \( (D^3, S^2) \). Let \( K \) be a \( CW \)-complex, and \( E = C(D^3, K), \ B = C(S^2, K) \) (the spaces of continuous maps). The map \( p : E \to B \) is defined as \( p(f : D^3 \to K) = (f|_{S^2} : S^2 \to K) \). Prove that the map \( p : E \to B \) is a Serre fiber bundle.

**Solution.** Let \( X = D^3, \ A = S^2 \). This is a \( CW \)-pair. Recall that a covering homotopy property holds for a map \( p : E \to B \), if for any \( CW \)-complex \( Z \) and commutative
and a homotopy $G : Z \times I \to B$, such that $G|_{Z \times \{0\}} = g$ there exists a homotopy $\tilde{G} : Z \times I \to E$ such that $\tilde{G}|_{Z \times \{0\}} = \tilde{g}$ and the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\tilde{g}} & Z \\
\downarrow{p} & & \downarrow{g} \\
B & & B
\end{array}
\]

commutes. Let $E = \mathcal{C}(X, K)$, $B = \mathcal{C}(A, K)$ be the spaces of continuous maps, and $p : E \to B$ as above. Let $Z$ be any CW-complex. Notice that there are natural homeomorphisms

\[
\begin{align*}
\mathcal{C}(Z, \mathcal{C}(X, K)) & \xrightarrow{\alpha_X} \mathcal{C}(Z \times X, K) \xrightarrow{\beta_X} \mathcal{C}(X, \mathcal{C}(Z, K)), \\
\mathcal{C}(Z, \mathcal{C}(A, K)) & \xrightarrow{\alpha_A} \mathcal{C}(Z \times A, K) \xrightarrow{\beta_A} \mathcal{C}(A, \mathcal{C}(Z, K)), \\
\mathcal{C}(Z \times I, \mathcal{C}(A, K)) & \xrightarrow{\alpha_A} \mathcal{C}(Z \times I \times A, K) \xrightarrow{\beta_{A \times I}} \mathcal{C}(A \times I, \mathcal{C}(Z, K))
\end{align*}
\]

given by the evaluation maps. Let $g : Z \to \mathcal{C}(A, K)$, $\tilde{g} : Z \to \mathcal{C}(X, K)$ be maps so that the diagram

\[
\begin{array}{ccc}
\mathcal{C}(X, K) & \xrightarrow{\tilde{g}} & Z \\
\downarrow{p} & & \downarrow{g} \\
\mathcal{C}(A, K) & & \mathcal{C}(A, K)
\end{array}
\]

commutes, and $G : Z \times I \to \mathcal{C}(A, K)$ be a homotopy, such that $G|_{Z \times \{0\}} = g$. We use the natural homeomorphisms $\alpha_A$, $\alpha_X$ to obtain the commutative diagram

\[
\begin{array}{ccc}
Z \times X & \xrightarrow{\alpha_X(g)} & K \\
\downarrow{Id \times i} & & \downarrow{Id} \\
Z \times A & \xrightarrow{\alpha_A(g)} & K
\end{array}
\]
where \( i : A \rightarrow X \) is the inclusion map. The homotopy \( G \) gives the map \( \alpha_{A \times I}(G) : Z \times A \times I \rightarrow K \) so that \( \alpha_{A \times I}(G)|_{Z \times A \times \{0\}} = \alpha_A(g) \). We use the homeomorphisms

\[
\begin{align*}
C(Z, C(X, K)) & \xrightarrow{\beta_x \alpha_X} C(X, C(Z, K)), \\
C(Z, C(A, K)) & \xrightarrow{\beta_{A \alpha A}} C(A, C(Z, K)), \\
C(Z \times I, C(A, K)) & \xrightarrow{\beta_{A \alpha A}} C(A \times I, C(Z, K))
\end{align*}
\]

to get the commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & C(Z, K) \\
\uparrow \downarrow_i & & \downarrow \downarrow_f \\
A & \xrightarrow{f \mid A} & C(A, K) \\
\uparrow \downarrow_{Id \times \{0\}} & & \downarrow \downarrow_F \\
A \times I & & \end{array}
\]

Since the pair \((A, X)\) is a \( CW\)-pair, and, in particular, Borsuk pair, there is a map \( \tilde{F} : X \times I \rightarrow C(Z, K) \) extending the homotopy \( F : A \times I \rightarrow C(Z, K) \). If we go back through the above homeomorphisms, we obtain that the homotopy \( G : Z \rightarrow C(A, K) \) lifts to a homotopy \( \tilde{G} : Z \rightarrow C(X, K) \). \( \square \)

3. Define the Hopf invariant \( h(\lambda) \) for \( \lambda \in \pi_{4q-1}(S^{2q}) \). Let \( \eta \in \pi_3(S^2) \) be given by the Hopf map \( S^3 \rightarrow S^2 \). Compute the Hopf invariant \( h(\eta) \).

**Solution.** Let \( \varphi \in \pi_{4n-1}(S^{2n}) \), and let \( f : S^{4n-1} \rightarrow S^{2n} \) be a representative of \( \varphi \). Let \( X_\varphi = S^{2n} \cup_f D^{4n} \). Compute the cohomology groups of \( X_\varphi \):

\[
H^n(X_\varphi; \mathbb{Z}) = \begin{cases} 
\mathbb{Z}, & n = 0, 2n, 4n, \\
0, & \text{otherwise.}
\end{cases}
\]

Let \( a \in H^{2n}(X_\varphi; \mathbb{Z}) \), \( b \in H^{4n}(X_\varphi; \mathbb{Z}) \) be generators. Since \( a^2 = a \cup a \in H^{4n}(X_\varphi; \mathbb{Z}) \), then \( a^2 = hb \), where \( h \in \mathbb{Z} \). The number \( h(\varphi) = h \) is the **Hopf invariant** of the element \( \varphi \in \pi_{4n-1}(S^{2n}) \).

Now consider the element \( \eta \in \pi_3(S^2) \) given by the Hopf map \( f : S^3 \rightarrow S^2 \). Then \( \mathbb{C}P^2 = S^2 \cup e^4 \), i.e., the Hopf map \( f \) is the attaching map for the cell \( e^4 \). Then \( H^*(\mathbb{C}P^2; \mathbb{Z}) \cong \mathbb{Z}[x]/x^3 \). Thus \( a = x \in H^2(\mathbb{C}P^2; \mathbb{Z}) \), and \( b = x^2 \in H^4(\mathbb{C}P^2; \mathbb{Z}) \), and \( a^2 = b \), so by definition \( h(\eta) = 1 \). \( \square \)

4. Let \( \alpha \in \pi_n(X), \beta \in \pi_k(X) \), and \( \Sigma : \pi_{n+k-1}(X) \rightarrow \pi_{n+k}(\Sigma X) \) be the suspension homomorphism. Prove that \([\alpha, \beta] \in \ker \Sigma\).

**Solution.** Consider the product \( S^n \times S^k \) as a \( CW\)-complex. Clearly we can choose a cell decomposition of \( S^n \times S^k \) into four cells of dimensions \( 0, n, k, n+k \). The first three cells give us the wedge \( S^n \vee S^k \subset S^n \times S^k \). The last cell \( e^{n+k} \subset S^n \times S^k \).
has the attaching map $w : S^{n+k-1} \to S^n \vee S^k$. This attaching map is called the \textbf{Whitehead map}. The map $w$ defines an element $w \in \pi_{n+k-1}(S^n \vee S^k)$. Denote $\iota_n \in \pi_n(S^n), \iota_k \in \pi_k(S^k)$ the generators given by the identity maps $\text{Id} : S^n \to S^n, \text{Id} : S^k \to S^k$ respectively. We denote also by $\iota_n, \iota_k$ the image of the elements $\iota_n, \iota_k$ in $\pi_n(S^n \vee S^k), \pi_k(S^n \vee S^k)$ respectively. Comparing the definitions of the Whitehead map $w : S^{n+k-1} \to S^n \vee S^k$ and of the Whitehead product gives the identity:

$$w = [\iota_n, \iota_k] \in \pi_{n+k-1}(S^n \vee S^k).$$

\textbf{Claim 0.1.} The element $w \in \pi_{n+k-1}(S^n \vee S^k)$ is in a kernel of each of the following homomorphisms:

1. $i_* : \pi_{n+k-1}(S^n \vee S^k) \to \pi_{n+k-1}(S^n \times S^k),$
2. $\text{pr}^{(n)}_* : \pi_{n+k-1}(S^n \vee S^k) \to \pi_{n+k-1}(S^n),$
3. $\text{pr}^{(k)}_* : \pi_{n+k-1}(S^n \vee S^k) \to \pi_{n+k-1}(S^k).$

\textbf{Proof.} The exact sequence

$$\to \pi_{n+k}(S^n \times S^k, S^n \vee S^k) \xrightarrow{\partial} \pi_{n+k-1}(S^n \vee S^k) \xrightarrow{i_*} \pi_{n+k-1}(S^n \times S^k) \to$$

implies that $w \in \text{Ker } i_*$ since $w = \partial(i)$. The commutative diagram

\[
\begin{array}{ccc}
\pi_{n+k-1}(S^n \vee S^k) & \xrightarrow{i_*} & \pi_{n+k-1}(S^n \times S^k) \\
\downarrow{\text{pr}^{(n)}_*} & & \downarrow{\text{pr}_.} \\
\pi_{n+k-1}(S^n) & & \\
\end{array}
\]

(where $\text{pr} : S^n \times S^k \to S^n$ is a map collapsing $S^k$ to the base point) implies that $w \in \text{Ker } \text{pr}^{(n)}_*$ and similarly $w \in \text{Ker } \text{pr}^{(k)}_*$. \hfill \Box

\textbf{Lemma 0.2.} There is an isomorphism

$$\pi_{n+k}(S^{n+1} \vee S^{k+1}) \cong \pi_{n+k}(S^{n+1}) \oplus \pi_{n+k}(S^{k+1})$$

\textbf{Proof.} Consider the long exact sequence for the pair $(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1})$:

$$\pi_{n+k+1}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) \xrightarrow{\partial} \pi_{n+k}(S^{n+1} \vee S^{k+1}) \xrightarrow{i_*} \pi_{n+k}(S^{n+1} \times S^{k+1})$$

$$\xrightarrow{j_*} \pi_{n+k}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) \to$$

We notice that the $(n + k + 1)$-skeleton of the product $S^{n+1} \times S^{k+1}$ is the wedge $S^{n+1} \vee S^{k+1}$. Thus any map $D^{k+n+1} \to S^{n+1} \times S^{k+1}$ may be deformed to the
subcomplex $S^{n+1} \vee S^{k+1}$. Thus $\pi_{n+k+1}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) = 0$. The same argument gives that

$$\pi_{n+k}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) = 0.$$

Thus the long exact sequence (2) gives the isomorphism:

$$i_* : \pi_{n+k}(S^{n+1} \vee S^{k+1}) \xrightarrow{\cong} \pi_{n+k}(S^{n+1} \times S^{k+1}) \cong \pi_{n+k}(S^{n+1}) \oplus \pi_{n+k}(S^{k+1}). \quad \square$$

**Claim 0.3.** The element $w \in \pi_{n+k-1}(S^n \vee S^k)$ is in the kernel of the suspension homomorphism

$$\Sigma : \pi_{n+k-1}(S^n \times S^k) \rightarrow \pi_{n+k}(\Sigma(S^n \times S^k)).$$

**Proof.** Consider the commutative diagram:

$$\begin{array}{cccc}
\pi_{n+k-1}(S^n) & \xrightarrow{pr^{(n)}_*} & \pi_{n+k-1}(S^n \vee S^k) & \xrightarrow{pr^{(k)}_*} & \pi_{n+k-1}(S^k) \\
\Sigma & & \Sigma & & \Sigma \\
\pi_{n+k}(S^{n+1}) & \xrightarrow{\Sigma(pr^{(n)}_*)} & \pi_{n+k}(\Sigma(S^n \vee S^k)) & \xrightarrow{\Sigma(pr^{(k)}_*)} & \pi_{n+k}(S^{k+1}) \\
\end{array}$$

(3)

where $pr$ denote the collapsing maps. By Claim 0.1 $w \in \text{Ker} \ pr^{(n)}_*$, $w \in \text{Ker} \ pr^{(k)}_*$. Notice that $\Sigma(S^n \vee S^k) \sim S^{n+1} \vee S^{k+1}$. Finally we notice that Lemma 0.2 and the diagram (3) imply that $w \in \text{Ker} \Sigma$. \quad \square

5. State the Lefschetz Fixed Point Theorem. Let $f : \mathbb{RP}^{10} \times \mathbb{RP}^{12} \rightarrow \mathbb{RP}^{10} \times \mathbb{RP}^{12}$ be a map. Prove that $f$ always has a fixed point.

**Solution.** First, the Lefschetz Fixed Point Theorem:

**Theorem 0.4.** Let $X$ be a finite CW-complex, $f : X \rightarrow X$ be a map such that $\text{Lef}(f) = 0$. Then $f$ has a fixed point, i.e. such point $x_0 \in X$ that $f(x_0) = x_0$.

Now the homology groups of the projective space $\mathbb{RP}^{2n}$ are as follows:

$$H_q(\mathbb{RP}^{2n}; \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & \text{if } q = 0, \\
\mathbb{Z}/2 & \text{if } q = 1, 3, \ldots, 2n - 1, \\
0 & \text{otherwise.}
\end{cases}$$

We use the Künneth formula to compute the homology groups $H_q(\mathbb{RP}^{2n} \times \mathbb{RP}^{2k}; \mathbb{Z})$:

$$0 \rightarrow \bigoplus_{r+s=q} H_r(X) \otimes H_s(X') \rightarrow H_q(X \times X') \rightarrow \bigoplus_{r+s=q-1} \text{Tor}(H_r(X), H_s(X')) \rightarrow 0$$

where $X = \mathbb{RP}^{2n}$, $X' = \mathbb{RP}^{2k}$. We notice that the groups $H_q(\mathbb{RP}^{2n} \times \mathbb{RP}^{2k}; \mathbb{Z})$ are all torsion groups unless $q = 0$. Indeed, $H_q(X') \rightarrow H_q(X \times X') = 0$ or
$\mathbb{Z}/2$ unless $s = r = 0$, and $\text{Tor}(H_r(X), H_s(X')) = 0$ or $\mathbb{Z}/2$ for all $s, r$. Thus $F(H_q(\mathbb{R}P^{2n} \times \mathbb{R}P^{2k}; \mathbb{Z})) = 0$ unless $q = 0$. Since the space $\mathbb{R}P^{2n} \times \mathbb{R}P^{2k}$ is path-connected, then any map $f : \mathbb{R}P^{2n} \times \mathbb{R}P^{2k} \to \mathbb{R}P^{2n} \times \mathbb{R}P^{2k}$ induces an isomorphism

$$f_* : \mathbb{Z} \cong H_0(\mathbb{R}P^{2n} \times \mathbb{R}P^{2k}; \mathbb{Z}) \to H_0(\mathbb{R}P^{2n} \times \mathbb{R}P^{2k}; \mathbb{Z}) \cong \mathbb{Z}.$$  

In particular, $\text{Tr}(f_*) \neq 0$. We obtain that $\text{Lev}(f) = \text{Tr}(f_*) \neq 0$. Then the Lefschetz Fixed Point Theorem implies that $f$ has a fixed point.

6. Give definition of weak homotopy equivalence. Prove that $\mathbb{R}P^n \times S^{n+1}$ is not homotopy equivalent to $\mathbb{R}P^{n+1} \times S^n$.

**Solution.** First, recall the definition. A map $f : X \to Y$ is a weak homotopy equivalence if for any CW-complex $Z$ the induced map $f_* : [Z, X] \to [Z, Y]$ is a bijection.

Now we consider the spaces $\mathbb{R}P^n \times S^{n+1}$ and $\mathbb{R}P^{n+1} \times S^n$. Assume that they are homotopy equivalent, then there exists a homotopy equivalence $f : \mathbb{R}P^n \times S^{n+1} \to \mathbb{R}P^{n+1} \times S^n$. In particular, the homology groups $H_q(X; \mathbb{Z}/2)$ and $H_q(Y; \mathbb{Z}/2)$ have to be isomorphic. We have (the coefficients are $\mathbb{Z}/2$):

$$H_r(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z}/2, & r = 0, \ldots, n, \\ 0 & \text{otherwise} \end{cases} \quad H_s(S^{n+1}) \cong \begin{cases} \mathbb{Z}/2, & s = 0, n + 1, \\ 0 & \text{otherwise} \end{cases}$$

$$H_r(\mathbb{R}P^{n+1}) \cong \begin{cases} \mathbb{Z}/2, & r = 0, \ldots, n + 1, \\ 0 & \text{otherwise} \end{cases} \quad H_s(S^n) \cong \begin{cases} \mathbb{Z}/2, & s = 0, n, \\ 0 & \text{otherwise} \end{cases}$$

In the case of $\mathbb{Z}/2$-coefficients, we have that

$$H_n(\mathbb{R}P^n \times S^{n+1}) = \bigoplus_{r+s=q} H_r(\mathbb{R}P^n) \otimes H_s(S^{n+1}) = H_n(\mathbb{R}P^n) \otimes H_0(S^{n+1}) = \mathbb{Z}/2;$$

$$H_n(\mathbb{R}P^{n+1} \times S^n) = \bigoplus_{r+s=q} H_r(\mathbb{R}P^{n+1}) \otimes H_s(S^n) = H_n(\mathbb{R}P^{n+1}) \otimes H_0(S^n) \oplus H_0(\mathbb{R}P^{n+1}) \otimes H_n(S^n) = \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$ 

Thus $H_n(\mathbb{R}P^n \times S^{n+1}) \not\cong H_n(\mathbb{R}P^{n+1} \times S^n)$, and these spaces are not homotopy equivalent.

7. State and prove the Jordan-Brouwer Theorem. (If you would like to use some preliminary results, please state them clearly.)

**Solution.** First, we state the Jordan-Brouwer Theorem:

**Theorem 0.5.** Let $S^{n-1} \subset S^n$ be an embedded sphere in $S^n$. Then the complement $X = S^n \setminus S^{n-1}$ has two path-connected components: $X = X_1 \cup X_2$, where $X_1, X_2$ are open in $S^n$. Furthermore, $\partial X_1 = \partial X_2 = S^{n-1}$. 

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Proof. We notice that
\[
\tilde{H}_q(S^n \setminus S^k) \cong \begin{cases} 
\mathbb{Z}, & \text{if } q = n - k - 1, \\
0, & \text{if } q \neq n - k - 1.
\end{cases}
\] (4)

The isomorphism (4) gives \( \tilde{H}_0(S^n \setminus S^{n-1}) \cong \mathbb{Z} \). Thus \( X = S^n \setminus S^{n-1} \) has two path-connected components: \( X = X_1 \sqcup X_2 \). It is clear that \( S^{n-1} \supset \partial X_1, S^{n-1} \supset \partial X_2 \). We have to prove that \( S^{n-1} \subset \overline{X_1} \cap \overline{X_2} \). It is enough to show that for any point \( x \in S^{n-1} \) and any open neighborhood \( U \) of \( x \) is \( S^n \), \( U \cap (\overline{X_1} \cap \overline{X_2}) \neq \emptyset \).

Let \( x \in S^{n-1} \), and \( U \) be an open neighborhood of \( x \) in \( S^n \). Now let \( B \subset S^{n-1}, B \subset U \), be a small neighborhood of \( x \) in \( S^{n-1} \), homeomorphic to an open disk \( \mathbb{D}^{n-1} \). Then the set \( A = S^{n-1} \setminus B \) is homeomorphic to the closed disk \( \mathbb{D}^{n-1} \). We have the following fact:

**Lemma 0.6.** Let \( K \subset S^n \) be homeomorphic to the cube \( I^k \), \( 0 \leq k \leq n \). Then \( \tilde{H}_q(S^n \setminus K) = 0 \) for all \( q \geq 0 \).

Thus \( \tilde{H}_q(S^n \setminus A) = 0 \) for all \( q \geq 0 \). In particular, it means that the subspace \( S^n \setminus A \) is path-connected. Now let \( p_1 \in X_1 \cap U \), and \( p_2 \in X_2 \cap U \). Then there exists a path \( \gamma : I \to S^n \setminus A \) connecting \( p_1 \) and \( p_2 \). Thus there exists \( t \in I \) so that \( \gamma(t) \in B \). Clearly \( p = \gamma(t) \) belongs to \( \overline{X_1} \cap \overline{X_2} \), and \( p \in S^{n-1} \). It means that \( U \cap (\overline{X_1} \cap \overline{X_2}) \neq \emptyset \). Since this is true for any open neighborhood \( U \) of \( x \), we obtain that \( x \in \overline{X_1} \cap \overline{X_2} \). \( \Box \)

8. Define a cup-product in cohomology. Let \( X = S^{2n} \times S^{2k} \). What is a ring structure of \( H^*(X; \mathbb{Z}) \). Explain.

**Solution.** First we need some notations. We identify a simplex \( \Delta^q \) with one given by its vertices \( (v_0, \ldots, v_q) \) in \( \mathbb{R}^{q+1} \). Let \( g : \Delta^q \to X \) be a map. It is convenient to use symbol \( (v_0, \ldots, v_q) \) to denote the singular simplex \( g : \Delta^q \to X \), and, say, \( (v_0, \ldots, v_s) \) the restriction \( g|_{(v_0, \ldots, v_s)} \).

Let \( R \) be a commutative ring with unit. We consider cohomology groups with coefficients in \( R \). Let \( \varphi \in C^k(X) \), \( \psi \in C^l(X) \) be singular cochains, and \( f : \Delta^{k+l} \to X \) be a singular simplex. We define the cochain \( \varphi \cup \psi \in C^{k+l}(X) \) as follows:
\[
(\varphi \cup \psi)(v_0, \ldots, v_{k+l}) = \varphi(v_0, \ldots, v_k)\psi(v_k, \ldots, v_{k+l}).
\]
To see that the cup-product at the level of cochains induces a product in cohomology groups, we have to understand the coboundary homomorphism on $\varphi \cup \psi$.

**Claim 0.7.** Let $\varphi \in C^k(X)$, $\psi \in C^l(X)$. Then

$$\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi.$$ 

Thus the cup-product is well-defined in cohomology:

$$\cup : H^k(X; R) \times H^l(X; R) \to H^{k+l}(X; R).$$

Then we define an **external cup product**

$$\mu : H^*(X_1; R) \otimes_R H^*(X_2; R) \to H^*(X_1 \times X_2; R)$$

as follows. Let $p_i : X_1 \times X_2 \to X_i$ ($i = 1, 2$) be the projection onto $X_i$, i.e. $p_i(x_1, x_2) = x_i$. Then $\mu(a \otimes b) = p_1^*(a) \cup p_2^*(b)$. The above tensor product $\otimes_R$ is taken over the ring $R$, where $H^*(X; R)$ are considered as $R$-modules. The tensor product $H^*(X_1; R) \otimes_R H^*(X_2; R)$ has natural multiplication defined as

$$(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) = (-1)^{\deg b_1 \deg a_2} (a_1 b_1 \otimes a_2 b_2).$$

**Claim 0.8.** The external product $H^*(X_1; R) \otimes_R H^*(X_2; R) \xrightarrow{\mu} H^*(X_1 \times X_2; R)$ is a ring homomorphism.

Moreover, we have the following fact:

**Theorem 0.9.** Let a space $X_2$ be such that $H^q(X_2; R)$ is finitely generated free $R$-module for each $q$. Then the external product

$$H^*(X_1; R) \otimes_R H^*(X_2; R) \xrightarrow{\mu} H^*(X_1 \times X_2; R)$$

is a ring isomorphism.

In the case when $X_1 = S^{2n}$, $X_2 = S^{2k}$, $R = \mathbb{Z}$, the conditions of the Theorem 0.9 are satisfied. We have that

$$H^*(S^{2q}; \mathbb{Z}) \cong \mathbb{Z}[\sigma_{2q}] / \sigma_{2q},$$

where $\sigma_{2q} \in H^{2q}(S^{2q}; \mathbb{Z})$ is a generator. We obtain that

$$H^*(S^{2n} \times S^{2k}) = \mathbb{Z}[\sigma_{2n}] / \sigma_{2n} \boxtimes \mathbb{Z}[\sigma_{2k}] / \sigma_{2k} = \Lambda_{\mathbb{Z}}(\sigma_{2n}, \sigma_{2k}).$$

9. Let $(X, A)$ be a CW-pair. Prove that the group $H^1(X, A; \mathbb{Z})$ is a free abelian group.
Solution. First we notice that $H_0(X, A; \mathbb{Z})$ is free abelian group. Then we use the universal coefficient formula

$$0 \to \text{Ext}(H_q(X, A; \mathbb{Z}), \mathbb{Z}) \to H^q(X, A; \mathbb{Z}) \to \text{Hom}(H_q(X, A; \mathbb{Z}), \mathbb{Z}) \to 0$$

for $q = 1$. The group $\text{Hom}(H_1(X, A; \mathbb{Z}), \mathbb{Z})$ is free abelian, and $\text{Ext}(H_0(X, A; \mathbb{Z}), \mathbb{Z}) = 0$ since $H_0(X, A; \mathbb{Z})$ is free abelian. □

10. Define a degree $\deg f$ of a map $f : S^n \to S^n$. Compute $\deg A$ of the antipodal map $A : x \mapsto -x$.

Solution. Let $f : S^n \to S^n$ be a map. Then the homotopy class $[f] = d\iota_n$, where $d \in \mathbb{Z}$, and $\iota_n \in \pi_n(S^n)$ is a generator represented by the identity map $S^n \to S^n$. Then, by definition, $\deg f := d$. We prove the following

**Lemma 0.10.** Let $A : S^n \to S^n$ be the antipodal map, $A : x \mapsto -x$, and $\iota_n \in \pi_n(S^n)$ be the generator represented by the identity map $S^n \to S^n$. Then the homotopy class $[A] \in \pi_n(S^n)$ is equal to

$$[A] = \begin{cases} \iota_n, & \text{if } n \text{ is odd}, \\ -\iota_n, & \text{if } n \text{ is even}. \end{cases}$$

**Proof.** Let $S^n$ be given as $x_1^2 + \cdots + x_{n+1}^2 = 1$ in $\mathbb{R}^{n+1}$. Let $n = 2k - 1$, then $n + 1 = 2k$. The antipodal map is given as $A : (x_1, \ldots, x_{2k}) \mapsto (-x_1, \ldots, -x_{2k})$.

Let $0 \leq \theta \leq \pi$. Consider the homotopy

$$A_\theta = \begin{pmatrix} \cos \theta & \sin \theta & \cdots & 0 & 0 \\ -\sin \theta & \cos \theta & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cos \theta & \sin \theta \\ 0 & 0 & \cdots & -\sin \theta & \cos \theta \end{pmatrix}$$

Then $A_0 = \text{Id}$, $A_\pi = A$. Thus $[A] = \iota_n$ if $n$ is odd.

The case $n = 2k$ is similar. We consider the homotopy

$$A_\theta = \begin{pmatrix} \cos \theta & \sin \theta & \cdots & 0 & 0 \\ -\sin \theta & \cos \theta & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cos \theta & \sin \theta \\ 0 & 0 & \cdots & -\sin \theta & \cos \theta \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Then $A_0 = -\iota_n$, and $A_\pi = A$. 

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