Analysis Qualifying Exam, Fall 2011

1. Let $C_0[0, 1]$ be the space of all continuous functions $f$ on $[0, 1]$ such that $f(0) = 0$, with the supremum $\| \cdot \|_{\infty}$ norm. Let
   \[ M = \{ f \in C_0[0, 1] : \int_0^1 f(t)dt = 1 \}. \]
   Prove that $M$ is a closed convex subset of $C_0[0, 1]$ which contains no element of minimal norm.

2. Prove that for any complex Lebesgue measurable function $f$ on $\mathbb{R}$ and $\varepsilon > 0$, there exists a continuous function $g : \mathbb{R} \to \mathbb{C}$ such that the set
   \[ \{ x \in \mathbb{R} : f(x) \neq g(x) \} \]
   has Lebesgue measure $< \varepsilon$.

3. Let $L^1[0, 1]$ be the space of all Lebesgue integrable functions on $[0, 1]$ with the usual $\| \cdot \|_1$ norm. Let $g$ be a Lebesgue measurable function on $[0, 1]$. Prove that $\|g\|_{\infty} < \infty \iff \int_0^1 |f(t)g(t)|dt < \infty$ for all $f \in L^1[0, 1]$.

4. Let $\mu$ be a complex Borel measure on $\mathbb{R}$. Suppose that $\{f_k\}_{k \in \mathbb{N}}$ is a collection of Borel functions on $\mathbb{R}$ such that
   \[ \sup_{k \in \mathbb{N}} \sup_{x \in \mathbb{R}} |f_k(x)| < \infty \quad \text{and} \quad f(x) = \lim_{k \to \infty} f_k(x) \text{ exists for all } x \in \mathbb{R}. \]
   Prove that
   \[ \lim_{k \to \infty} \int f_k d\mu = \int f d\mu. \]

5. Suppose that $f : [a, b] \to \mathbb{R}$ is AC (absolutely continuous). Prove that there exist non-decreasing AC functions $f_1, f_2 : [a, b] \to \mathbb{R}$ such that $f = f_1 - f_2$.

6. Let $c_0$ consists of all sequences $x = (x_k)_{k \in \mathbb{N}}$ such that $x_k \to 0$ as $k \to \infty$ with the norm
   \[ \|x\|_{\infty} = \sup_{k \in \mathbb{N}} |x_k|. \]
   Prove that the dual space $(c_0)^*$ can be identified with the space $\ell^1$, which consists of all summable sequences on $\mathbb{N}$.

7. Let $f \in L^1(\mathbb{R})$ and $g = \chi_{[0,1]}$. Suppose that their convolution
   \[ f * g(x) = 0 \quad \text{for a.e. } x \in \mathbb{R}. \]
   Prove that $f = 0$ a.e.

8. Let $f$ be a holomorphic function on the open disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$. Suppose that $\|f\|_{\infty} = \sup_{z \in D} |f(z)| < 1$. Prove that $f$ has exactly one fixed point $z_0 \in D$, i.e., $f(z_0) = z_0$.

9. Let $X$ be the closure of the set $\{ 1/n + i/m : n, m \in \mathbb{N} \}$. Suppose that $f$ is a holomorphic function on $\mathbb{C} \setminus X$. Prove that if $f$ is bounded on $\{ z \in \mathbb{C} \setminus X : |z| < 3 \}$, then $f$ extends to an entire function.