Analysis Qualifying Exam, Fall 2006

1. Suppose that $\mu$ is a positive Borel measure on $\mathbb{R}^2$ such that
\[ \mu(L) > 0 \quad \text{for any straight line } L \subset \mathbb{R}^2. \]
Prove that $\mu(\mathbb{R}^2) = \infty$.

2. Give an example of a nowhere dense set $E \subset [0, 1]$ such that $|E| > 0$.

3. Let $f$ be a real-valued Lebesgue measurable function on $\mathbb{R}^n$. Prove that there exists a Borel function $g$ on $\mathbb{R}^n$ such that $f(x) = g(x)$ for a.e. $x \in \mathbb{R}^n$.

4. Let $f \in L^1(\mathbb{R})$ be compactly supported. Prove that
\[ g(z) = \int_{-\infty}^{\infty} f(x) e^{xz} \, dx \]
is an entire function. Conclude that $\hat{f}(\xi)$ has at most countably many zeros.

5. Let $\mathcal{H}$ be a Hilbert space. We say that a sequence $\{x_n\} \subset \mathcal{H}$ converges weakly to $x$ if
\[ \langle x_n, y \rangle \to \langle x, y \rangle \quad \text{for all } y \in \mathcal{H}. \]
Suppose that $\{T_n\}$ is a sequence of bounded linear operators on $\mathcal{H}$ such that for all $x \in \mathcal{H}$, $\{T_n x\}$ converges weakly to some $y = Tx$. Prove that $T$ is a bounded linear operator on $\mathcal{H}$.

6. Suppose that $\Lambda$ is a bounded linear functional on $C_0(\mathbb{R})$. Prove that the limit
\[ \lim_{n \to \infty} \Lambda(f_n), \quad f_n(x) = e^{-n|x|} \]
eexists. Describe when it is non-zero.

7. Let $f = \chi_{[0,1]}$. Prove that its maximal function
\[ Mf(x) = \sup_{I \text{ interval s.t. } x \in I} \frac{1}{|I|} \int_I |f| \]
is not in $L^1(\mathbb{R})$.

8. Suppose that $f$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and there exists $N \in \mathbb{N}$ such that
\[ |f(z)| \leq C|z|^{-N} \quad \text{for all } z \neq 0. \]
Prove that $f$ is a rational functional with a pole at 0 of order at most $N$.

9. Let $C = \{z \in \mathbb{C} : |z| = 1\}$. Suppose that $f$ is a holomorphic function on $\mathbb{C} \setminus C$ and continuous on $\mathbb{C}$. Prove that $f$ is entire.