Part I. Definitions and Theorems.

1 (6 points). Give two equivalent definitions of projective module.
2 (6 points). State the Primitive Element Theorem from the theory of field extensions.

Part II. True or false. Give brief justification.

1 (8 points). The center of a nontrivial solvable group is nontrivial.
2 (8 points). Let $f(x)$ be an irreducible separable polynomial of degree $n$. Then $|Gal(f(x))|$ is divisible by $n$.
3 (8 points). Let $G$ be a finite group such that the group $Aut(G)$ is cyclic. Then $G$ is abelian.
4 (8 points). The ring $C[x, y]$ is Jacobson semisimple.
5 (8 points). Let $R = F_{25}[x]$. Then $K_0(R) = Z$.

Part III. Longer problems.

You have to solve any 4 of the problems below.

1 (12 points). Let $I$ and $J$ be two ideals of $C[x_1, \ldots, x_n]$. Assume that $IJ$ is radical. Prove that $IJ = I \cap J$.
2 (12 points). Let $A$ be a noncommutative finite dimensional algebra over $C$ such that the length of $A$–module $A$ is 2. What is $dim(A)$?
3 (12 points). Find the Galois group of the polynomial $x^4 + x^2 + 1$ over $Q$.
4 (12 points). Let $V$ be a linear space of dimension $n$ and $F : V \rightarrow V$ be a linear operator with determinant $D$. What is determinant of $\bigwedge^2 F : \bigwedge^2 V \rightarrow \bigwedge^2 V$?
5 (12 points). Let $G$ be a finite group with precisely 5 inequivalent irreducible representations of dimension 1,1,2,3 and $d$. Find $d$. 
Part I. Definitions and Theorems.

1 (6 points). Give two equivalent definitions of projective module.

**Answer:** $P$ is projective if

1) for any exact sequence $A \rightarrow^{f} B \rightarrow 0$ and a homomorphism $f : P \rightarrow B$ there exists a homomorphism $g : P \rightarrow A$ such that $f = \phi \circ g$.
2) The functor $\text{Hom}(P, ?)$ is exact.
3) $P$ is a direct summand of a free module.

2 (6 points). State the Primitive Element Theorem from the theory of field extensions.

**Answer:** A finite separable extension $E/K$ can be generated by one element: $E = K(\alpha)$.

Part II. True or false. Give brief justification.

1 (8 points). The center of a nontrivial solvable group is nontrivial.

**Solution.** False. Group $S_3$ is solvable with trivial center.

2 (8 points). Let $f(x)$ be an irreducible separable polynomial of degree $n$. Then $|\text{Gal}(f(x))|$ is divisible by $n$.

**Solution.** True. The action of $\text{Gal}(f(x))$ on the roots of $f(x)$ is transitive, hence $|\text{Gal}(f(x))| = n \cdot |St|$ where $St \subset \text{Gal}(f(x))$ is a stabilizer of a root.

3 (8 points). Let $G$ be a finite group such that the group $\text{Aut}(G)$ is cyclic. Then $G$ is abelian.

**Solution.** True. Group $G/Z(G)$ is a subgroup of $\text{Aut}(G)$ (inner automorphisms). A subgroup of a cyclic group is cyclic and thus $G/Z(G)$ should be cyclic. Thus $G/Z(G)$ is trivial.

4 (8 points). The ring $\mathbb{C}[x, y]$ is Jacobson semisimple.

**Solution.** True. The Jacobson radical is an intersection of all maximal ideals. By Nullstellensatz the maximal ideals are of the form $\{f \in \mathbb{C}[x, y] \mid f(a, b) = 0\}$ for various $(a, b) \in \mathbb{C}^2$. Clearly, the intersection is zero.

5 (8 points). Let $R = F_{25}[x]$. Then $K_0(R) = \mathbb{Z}$.

**Solution.** True. The ring $R$ is PID, hence any finitely generated projective module is free; free modules over domain are (stably) isomorphic only if ranks coincide.

Part III. Longer problems.
You have to solve any 4 of the problems below.

1 (12 points). Let \( I \) and \( J \) be two ideals of \( \mathbb{C}[x_1, \ldots, x_n] \). Assume that \( IJ \) is radical. Prove that \( IJ = I \cap J \).

**Solution.** We have \( IJ \subset I \cap J \subset \sqrt{I \cap J} = \sqrt{IJ} = IJ \). The result follows.

2 (12 points). Let \( A \) be a noncommutative finite dimensional algebra over \( \mathbb{C} \) such that the length of \( A \)-module \( A \) is 2. What is \( \text{dim}(A) \)?

**Solution.** First assume that \( J(A) \neq 0 \). Then \( J(A) \) and \( A/J(A) \) should be simple \( A \)-modules. Thus \( A/J(A) \) is simple \( A/J(A) \)-module and hence \( A/J(A) = \mathbb{C} \). Since \( J(A) \) is simple over \( A/J(A) \) it is one dimensional over \( \mathbb{C} \). Thus \( A \) is two dimensional and hence commutative. Thus \( J(A) = 0 \) and \( A \) is semisimple. We have two possibilities: \( A = \mathbb{C} \oplus \mathbb{C} \) and \( A = \text{Mat}_2(\mathbb{C}) \). First is again commutative. Thus \( A = \text{Mat}_2(\mathbb{C}) \).

**Answer:** \( \text{dim}(A) = 4 \).

3 (12 points). Find the Galois group of the polynomial \( x^4 + x^2 + 1 \) over \( \mathbb{Q} \).

**Solution.** We have \( x^4 + x^2 + 1 = (x^2 + 1)^2 - x^2 = (x^2 - x + 1)(x^2 + 1) \). The roots of \( x^2 - x + 1 \) and \( x^2 + 1 \) differ only by sign, hence generate the same quadratic field. Hence \( \text{Gal}(x^4 + x^2 + 1) = \text{Gal}(x^2 - x + 1) = \mathbb{Z}/2\mathbb{Z} \).

**Answer:** \( \text{Gal}(x^4 + x^2 + 1) = \mathbb{Z}/2\mathbb{Z} \).

4 (12 points). Let \( V \) be a linear space of dimension \( n \) and \( F : V \to V \) be a linear operator with determinant \( D \). What is determinant of \( \Lambda^2 F : \Lambda^2 V \to \Lambda^2 V \)?

**Solution.** We can assume that the base field is algebraically closed. Using Jordan normal form we can choose a basis \( \{e_1, \ldots, e_n\} \) such that \( F \) is upper triangular with eigenvalues \( \lambda_1, \ldots, \lambda_n \). Now in basis \( e_1 \wedge e_2, \ldots, e_1 \wedge e_n, e_2 \wedge e_3, \ldots, e_{n-1} \wedge e_n \) the operator \( \Lambda^2 F \) is upper triangular with eigenvalues \( \lambda_1 \lambda_2, \ldots, \lambda_{n-1} \lambda_n \). Hence determinant of \( \Lambda^2 F \) is \( \prod_{i<j} \lambda_i \lambda_j = (\prod_i \lambda_i)^{n-1} = D^{n-1} \).

**Answer:** \( D^{n-1} \).

5 (12 points). Let \( G \) be a finite group with precisely 5 inequivalent irreducible representations of dimension 1, 1, 2, 3 and \( d \). Find \( d \).

**Solution.** We have \( |G| = 1^2 + 1^2 + 2^2 + 3^2 + d^2 = 15 + d^2 \). Since \( |G| \) should be divisible by 2, 3 and \( d \) we find that \( d = 3 \) or \( d = 15 \). Assume that \( d = 15 \) and \( |G| = 240 \). Then \( |G| : G'| = 2 \) (since \( G \) has precisely 2 linear characters) and \( G' \) contains at least 4 conjugacy classes of \( G \): elements of order 1, 2, 3, 5. Thus \( G - G' \) should be a single conjugacy class of size 120. Thus for \( x \in G - G' \) we have \( |C_G(x)| = 2 \). This implies \( x^2 = 1 \) (since \( 1, x, x^2 \in C_G(x) \)) and hence \( x \) lies in a Sylow 2-subgroup \( P \). But then \( |C_G(x)| > 2 \): either \( x \) is not central in \( P \) and then \( C_G(x) \) contains \( x \) and \( Z(P) \), or \( x \) is central and \( C_G(x) \) contains \( P \). This is a contradiction. Hence \( d = 3 \) (this is really possible; \( G = S_4 \) is an example).

**Answer:** \( d = 3 \).
You have to solve any 4 of the problems below.

1 (12 points). Let \( I \) and \( J \) be two ideals of \( \mathbb{C}[x_1, \ldots, x_n] \). Assume that \( IJ \) is radical. Prove that \( IJ \subseteq I \cap J \).

Solution. We have \( IJ \subseteq I \cap J \subseteq \sqrt{IJ} = IJ \). The result follows.

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Answer: \( \dim(A) = 4 \).

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Answer: \( D^{n-1} \).

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Answer: \( d = 3 \).