Assume all rings have identity elements.

**Section 1:** State the theorems below, defining relevant terms.
   i. The Jacobson density theorem.
   ii. The theorem describing a matrix in terms of its rational canonical form.
   iii. The fundamental theorem of Galois theory.

**Section 2:** True/False. If FALSE, provide a counterexample, if TRUE, give a brief justification.
   a. If $R$ is a commutative ring, any submodule of a free module is itself free.
   b. Every group has a non-trivial center.
   c. A finite linear transformation $L$ on a vector space $V$ of characteristic 0 is nilpotent if and only if the trace of $L$ is 0.
   d. Any decreasing sequence of varieties of $K^n$, $V_1 \supseteq V_2 \supseteq \ldots$ stabilizes.
   e. $R$ is simple if and only if $R \cong \text{Mat}_n(D)$ for $D$ a division algebra.
   f. If $R$ is commutative, and every submodule of a free module is free, then $R$ is a P.I.D.

**Section 2:** Give complete proofs for 4 problems from the following.

1. Prove that if $0 \to A \to B \to C \to 0$ is a short exact sequence of left $R$-modules then $A, C$ Noetherian if and only if $B$ is Noetherian.
2. Prove that a ring $R$ with 1 has orthogonal central idempotents $e_1, \ldots, e_n$ such that
   \[ 1 = e_1 + \cdots + e_n \]
   if and only if
   \[ R \cong A_1 \times A_2 \times \cdots \times A_n \]
   for some principal ideals $A_1, \ldots, A_n$.
3. Work in the category of abelian groups. Prove that the cartesian product $A \times B$ is both a categorical product and a categorical sum.
4. Let $R$ be a P.I.D.
   a. Which cyclic $R$ modules are projective?
   b. Calculate $\text{Hom}_R(R/(a), R/(b))$.
5. a. State the Hilbert basis theorem.
   b. Prove that if $S$ is a finitely generated (as an algebra) commutative ring extension of $K$, then $S$ is Noetherian.
Assume all rings have identity elements.

Section 1: State the theorems below, defining relevant terms.

  i. The Jacobson density theorem.

    If $R$ is a primitive ring with faithful simple $R$-module $A$, then $A$ can be considered as a vector space over the division ring $\text{Hom}_R(A, A)$. $R$ is isomorphic to a dense ring of $D$-endomorphisms of $A$.

    $A$ is left faithful if no elements of $R$ (besides 0) annihilate all of $A$. $R$ is left primitive if it has a left faithful simple $R$-module. An $R$-module is left-simple if it has no proper left sub-modules.

    $R$ is dense if for each (finite) linearly independent $a_1, \ldots, a_n \in A$ and each set of elements $s_1, \ldots, s_n \in A$ there is an element of $R$ so that $r(a_i) = s_i$.

  ii. The theorem describing a matrix in terms of its rational canonical form.

    Let $L : V \rightarrow V$ be an $n \times n$ matrix over $K$. $V$ has a basis that make $L$ a direct sum of its companion (to the factors of the minimal polynomial) matrices.

    The companion matrix $q_i$ describes $L$ acting on a cyclic subspace. The factor $q_i$ of the minimal polynomial is a generator of the ideal for which $K[x]/(q_i(x))$ is isomorphic to the corresponding summand of $V$, and in rational canonical form, a basis has been chosen with 1s along the diagonal before lunch, and the coefficients of $q_i(x)$ along the right-hand side and 0s elsewhere.

  iii. The classifications of finitely generated modules over a P.I.D.

    If $M$ is finitely generated over a P.I.D. $R$, $M \cong R^n \oplus R/(p_1^{m_1}) \oplus \cdots \oplus R/(p_k^{m_k})$ where $p_i$ is a prime element of $R$ and $r_i$ is an integer. The integer $m_i$ and the ideals $(p_i^{m_i})$ are uniquely determined (except for order).

    Under the same hypotheses, $M \cong R^n \oplus R/(d_1) \oplus R/(d_2) \oplus \cdots \oplus R/(d_r)$ where $d_1 | d_2 | \cdots | d_r$. $m$ and the ideals $(d_i)$ are uniquely determined by $M$ (up to order).

Section 2: True/False. If FALSE, provide a counterexample, if TRUE, give a brief justification.

  a. If $R$ is a commutative ring, any submodule of a free module is itself free.

    FALSE. Let $R = \mathbb{Z}[x, y]$. Then the submodule $(x, y) \subseteq R$ is not free.

  b. Every group has a non-trivial center.

    FALSE. Any simple group (e.g. $A_n$ for $n \geq 5$) has no non-identity central elements, else the center would form a non-trivial normal subgroup.

  c. A finite linear transformation $L$ on a vector space $V$ of characteristic 0 is nilpotent if and only if the trace of $L$ is 0.

    FALSE. Consider

    $$L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

d. Any decreasing sequence of varieties of $K^n$, $V_1 \supseteq V_2 \supseteq \cdots$ stabilizes.

    TRUE. The sequence of varieties corresponds to a sequence of ideals

    $$I(V_1) \subseteq I(V_2) \subseteq I(V_3) \subseteq \cdots$$

    in $K[x_1, \ldots, x_n]$, which is Noetherian, so this sequence of ideals stabilizes.

  e. $R$ is simple if and only if $R \cong \text{Mat}_n(D)$ for $D$ a division algebra.

    FALSE. The example we discussed in class and had on homework, $R = K[x, y]/(yx = xy + 1)$. A laborious demonstration involving expressing all monomials beginning with $x$ can demonstrate simplicity.

    If $R$ were a division ring, denote $x^{-1}$ by $h$. Then $xh = hx = 1$. This implies no powers of $y$ are present in $h$. Then $h$ would be an inverse for $x$ in $F[x]$. But this is nonsense.
f. If \( R \) is commutative, and every submodule of a free module is free, then \( R \) is a P.I.D.

TRUE. Every ideal of \( R \) is free, so if an ideal is a free module on more than one generator, let \( u, v \) be two generators. Then \( v \cdot u - u \cdot v = 0 \) which gives a relation involving the generators. Hence the ideal can't be free. So the ideal must be free on a single generator, hence principal.

Section 2: Give complete proofs for 4 problems from the following.

(1) Prove that if \( 0 \to A \to B \to C \to 0 \) is a short exact sequence of left \( R \)-modules then \( A, C \) Noetherian if and only if \( B \) is Noetherian.

First we do the "only if." Given a chain of submodules of \( B \),

\[ B_1 \subseteq B_2 \subseteq \ldots \]

we note that \( B_i/(A \cap B_i) \) stabilizes since \( C \) Noetherian. WLOG assume it is stable immediately - we can always remove the first finitely many \( B_i \).

Also since \( A \) Noetherian, \( A \cap B_i \) stabilizes. This implies \( B_i \) stabilizes.

Now the "if." If we assume \( B \) is Noetherian, so is \( A \) since it is a submodule. If \( C_1 \subseteq C_2 \subseteq \ldots \) is an increasing chain in \( C \), then we look at the inverse images of these modules in \( B \). They stabilize, so the \( C_i \) (which are the images of the inverse images) stabilize also.

(2) Prove that a ring \( R \) with 1 has orthogonal central idempotents \( e_1, \ldots, e_n \) such that

\[ 1 = e_1 + \cdots + e_n \]

if and only if

\[ R \cong A_1 \times A_2 \times \cdots \times A_n \]

for some principal ideals \( A_1, \ldots, A_n \).

Suppose we have a set of orthogonal central idempotents \( e_1, \ldots, e_n \) such that

\[ 1 = e_1 + \cdots + e_n. \]

Take \( A_i = e_i R \). Since \( e_i \) is central, this is a two-sided ideal. Since \( 1 = e_1 + \cdots + e_n \),

\[ R = A_1 + \cdots + A_n. \]

Since \( e_i e_j = 0 \) if \( i \neq j \), we have

\[ e_i(e_1 r_1 + \cdots + e_{i-1} r_{i-1} + e_{i+1} r_{i+1} + \cdots + e_n r_n) = 0 \]

so if \( x \in A_i \cap (A_1 + \cdots + A_{i-1} + A_{i+1} + \cdots + A_n) \) then \( x = e_i x = 0 \).

So a theorem about the decomposition of rings gives us

\[ R \cong A_1 \times \cdots \times A_n \]

Conversely, suppose

\[ R \cong A_1 \times \cdots \times A_n \]

for ideals \( A_1, \ldots, A_n \). Keep in mind that the isomorphism from right-to-left is given by

\[ f(a_1, \ldots, a_n) \mapsto a_1 + a_2 + \cdots + a_n \]

and that this isomorphism defines projections \( \pi_i : R \to A_i \) by taking \( x \) to \( f^{-1}(x) \) and then projecting to the \( i \)th coordinate.

Take \( e_i = \pi_i(1) \in A_i \subseteq R \). Then \( 1 = e_1 + \cdots + e_n \), and \( e_i = \pi_i(1) = \pi_i(1^2) = (\pi_i(1))^2 = e_i^2 \).

By our isomorphism, \( e_i e_j \in R \) is the sum of the coordinates of \((0, \ldots, e_i, 0, \ldots)\) \((0, \ldots, e_j, \ldots) \) where \( e_i \) is in the \( i \)th coordinate and \( e_j \) is in the \( j \)th coordinate. But this is 0.
Finally, $e_i$ is central in $A_i$ since it is the image of a central element, 1, and since $\pi_i$ is onto. It is central in the product since $(0, \ldots, 0, e_i, 0, \ldots 0)$ times an element in another coordinate is 0.

(3) Work in the category of abelian groups. Prove that the cartesian product $A \times B$ is both a categorical product and a categorical sum.

Suppose we have maps of abelian groups $f : A \to C$ and $g : B \to C$. We define maps $i_A : A \to A \times B$ by $i_A(a) = (a, 0)$ and similarly for $i_B$.

Then we define $F : A \times B \to C$ by $F(a, b) = f(a) + g(b)$. Since $(a, b) = (a, 0) + (b, 0) = i_A(a) + i_B(b)$ this is the only possible way to define $F$ so that $F \circ i_A = f$ and $F \circ i_B = g$, and is a well-defined group homomorphism from the abelian group $A \times B \to C$.

Now define $\pi_A : A \times B \to A$ by $\pi_A(a, b) = a$ and similarly for $\pi_B$.

Given maps $f : C \to A$ and $g : C \to B$, define $F : C \to A \otimes B$ by $F(c) = (f(c), g(c))$. It is just as easy to check this satisfies the universal conditions for a product as the previous case.

(4) Let $R$ be a P.I.D.

(a) Which cyclic $R$ modules are projective?

$R$ itself is projective. If $a \neq 0$ then $R/(a)$ contains torsion. So it is not a submodule of any free module, thus not a summand of any free module, hence not projective. So $R$ is the only cyclic module that is projective.

(b) Calculate $\text{Hom}_R(R/(a), R/(b))$.

The generator 1 of $R/(a)$ must go to some element of $R/(b)$ which is annihilated by $a$. If $b$ is prime to $a$ the only such element is 0 by our familiar fact that $ua + vb = 1$ for some $u, v$.

If $d$ is the G.C.D. of $a$ and $b$ then $b = kd$ and 1 must go to a multiple of $k$ in $R/(b)$. So the $\text{Hom}$ set is $R/(d)$ generated by $k$ in $R/(b)$.

(5) (a) State the Hilbert basis theorem.

(b) Prove that if $S$ is a finitely generated (as an algebra) commutative ring extension of $K$, then $S$ is Noetherian.

If $R$ is a commutative Noetherian ring with 1, then so is $R[x_1, \ldots, x_n]$. Under the hypothesis, $S = K[u_1, \ldots, u_n]$ for some $u_i \in S$ (not the polynomial ring, but the smallest subring of $S$ containing the $u_i$). So $S$ is a quotient of $K[x_1, \ldots, x_n]$. $K[x_1, \ldots, x_n]$ is Noetherian ring by the Hilbert basis theorem, and $S$ is a quotient, so $S$ is a Noetherian $K[x_1, \ldots, x_n]$-modules, hence $S$ is a Noetherian $S$-module.