ALGEBRA QUALIFYING EXAM, FALL 2003

Conventions: All rings have identity, and all modules are unital.

Part I. Carefully state each of the following (be certain that any special terminology or notation is explained):

1. Hilbert Nullstellensatz.
2. Sylow Theorems.

Part II. For each of the following determine whether it is true or false. If true, give a brief explanation. If false, provide a counterexample:

1. If $G$ is a group such that any finitely generated subgroup of $G$ is cyclic, then $G$ is cyclic.
2. If $G$ is a finite nilpotent group, and $m$ is a positive integer dividing $|G|$, then there exists a subgroup of $G$ of order $m$.
3. If $F$ is an infinite field, then its multiplicative group $F^\times$ is never cyclic.
4. Let $R$ be a noetherian local ring with maximal ideal $M$ and let $x_1, \ldots, x_n \in M$. If $\{x_1 + M^2, \ldots, x_n + M^2\}$ is a basis of the $R/M$-vector space $M/M^2$ then $M = Rx_1 + \cdots + Rx_n$.
5. If $R$ is a noetherian commutative ring, then every $R/J(R)$-module is semisimple.
6. Let $F$ be an algebraically closed field of characteristic $p > 0$, and $f : F \to F, a \mapsto a^p$ be the Frobenius homomorphism. True or false:
   (a) $f$ is a homeomorphism in the Zariski topology.
   (b) $f$ is an isomorphism of algebraic sets.

Part III. Give complete solutions for each of the following problems.

1. Let $G$ be a finite group. We choose an element $g \in G$ randomly. Then replace it and make another random choice of an element $h \in G$. Prove that the probability that $g$ and $h$ commute equals to $k/|G|$, where $k$ is the number of conjugacy classes in $G$.
2. Let $p$ be a prime. Then there are exactly $(q^p - q)/p$ monic irreducible polynomials of degree $p$ in $\mathbb{F}_q[x]$ ($q$ is not necessarily a power of $p$).
3. Let $G$ be a finite group and $F$ be an algebraically closed field of characteristic $p \geq 0$. Prove the following:
   (i) Up to isomorphism, that there are only finitely many irreducible $FG$-modules $L_1, \ldots, L_k$. 
(ii) Let $d_i = \dim L_i$, $1 \leq i \leq k$. Then $\sum_{i=1}^{k} d_i^2 \leq |G|$, and the equality holds if and only if $p_i | |G|$. 

(iii) Is it true that the inequality $\sum_{i=1}^{k} d_i^2 \leq |G|$ holds even if $F$ is not algebraically closed?

4. Let $R$ be a ring and $V_1, V_2$ be non-isomorphic simple $R$-modules. Prove that $V_1 \oplus V_2$ has exactly four submodules: $0, 0 \oplus V_2, V_1 \oplus 0$, and $V_1 \oplus V_2$.

5. Let $V$ and $W$ be vector spaces over the same field $F$ of finite dimensions $n$ and $m$, respectively. Assume that $f : V \rightarrow V$ and $g : W \rightarrow W$ are linear transformations. Prove that $\det f \otimes g = (\det f)^m(\det g)^n$. 