THE LAST HOMEWORK. DUE FRIDAY JUNE 6

HAND IN:

1. (i) If $K \subseteq L$ is a normal extension and $Aut_K(L) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, prove that L is a splitting field of a polynomial $p(x) = (x^2 - a)(x^2 - b)$ for some $a, b \in K$. (FTGT)

(ii) If $K \subseteq L$ is a normal extension and $Aut_K(L) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, prove that L is a splitting field of a polynomial $p(x) = (x^2 - a)(x^2 - b)(x^2 - c)$ for some $a, b, c \in K$.

2. Let $f = (x^3 - 2)(x^3 - 3) \in \mathbb{Q}[x]$ and denote $G = Gal(f)$.

(i) Find $|G|$.

(ii) True of False and Why: G is abelian.

(iii) True of False and Why: G is solvable.

3. Let $E \supset F$ be a normal extension whose degree is a power $p^k > 1$ of a prime p.

(i) Prove: E has a subfield that is a normal extension of F of degree p^{k-1} (Hint: p-groups from the second term).

(ii) Prove: E has a subfield that is a normal extension of F of degree p^i whenever $1 \leq i \leq k$.

4. Let $E(f)$ be the splitting field of a nonconstant polynomial $f \in K[x]$ with no multiple roots. Prove: f is irreducible over K if and only if $Aut_K(L)$ is transitive on the set of roots of f.

5* (**bonus problem**). Let $n > 0$ be a natural number and $\zeta \in \mathbb{C}^\times$ be a primitive n-th root of unity).

(i) Prove that $Aut_{\mathbb{Q}}(\mathbb{Q}(\zeta)) \cong \mathbb{Z}_n^\times$.

(ii) Prove that the minimal polynomial $\Phi_n(x) \in \mathbb{Q}[x]$ of ζ factors as

$$\Phi_n(x) = \prod (x - \zeta^k) \ ,$$

where the product is over all naturals $k < n$ such that $gcd(k, n) = 1$ (it is called the n-th *cyclotomic polynomial*)

6* (**bonus problem**). Let $a = \frac{m}{n} \neq 0$ be any rational number such that: $m \in \mathbb{Z}$, $m \notin 5\mathbb{Z}$, $n \in \mathbb{N}$, $n \notin 5\mathbb{N}$ and $|a| < \frac{4}{5} \cdot \sqrt[4]{\frac{1}{5}}$. Prove that the equation $x^5 - x - a = 0$ is NOT solvable in radicals over $\mathbb{Q}$.