HOMEWORK 7-8. DUE FRIDAY MAY 23

HAND IN: 8.2.2, 8.5.1, 8.6.8, 8.6.15* (**bonus problem**), 8.7.1, 8.7.3.

1. Let n be a positive integer and $\zeta \in \mathbb{C}^\times$ be of order n (such a ζ is called a *primitive n-th root of unity*).
   (i) Prove: $\mathbb{Q}(\zeta)$ is the splitting field of $x^n - 1$.
   (ii)*(**bonus question**) Prove: $Aut_{\mathbb{Q}}(\mathbb{Q}(\zeta))$ is abelian.

2. Let $L \subseteq \mathbb{C}$ be the splitting field of $x^p - 2 \in \mathbb{Q}[x]$, where p is a prime.
   (i) Prove: $L = \mathbb{Q}(\sqrt[p]{2}, \zeta_p)$.
   (ii) Deduce: $\dim_{\mathbb{Q}} L \leq p(p-1)$.
   (iii) Prove: $\dim_{\mathbb{Q}} L = p(p-1)$.
   (iv) Prove: $x^p - 2 \in \mathbb{Q}(\zeta_p)[x]$ is irreducible.

3. (i) Prove: $L = \mathbb{Q}(\sqrt[4]{3}, i)$ is the splitting field of $x^4 - 3$ over $\mathbb{Q}$.
   (ii) Prove: $dim_{\mathbb{Q}}(L) = 8$. ($\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[4]{3}) \subseteq L$.)
   (iii) Prove: Some $\sigma \in Aut_{\mathbb{Q}}(L)$ sends $\sqrt[4]{3} \to i\sqrt[4]{3}$ and $i \to i$.
   (iv) Prove: If τ denotes the restriction to L of complex conjugation, then $Aut_{\mathbb{Q}}(L) = \{1, \sigma, \sigma^2, \sigma^3, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\} \cong D_4$. (Why are these elements of $Aut_{\mathbb{Q}}(L)$ different?)

4. For $F = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$:
   (i) Prove that $G = Aut_{\mathbb{Q}}(F)$ has order 8.
   (ii) Write down all 8 subgroups of G, and for each identify the fixed field as a simple extension of $\mathbb{Q}$. (There are 7 subgroups of order 2 and 7 of order 4.)
   (iii) Exhibit the Galois correspondence for F.

DO NOT HAND IN: 8.1.1, 8.1.2, 8.1.3, 8.2.3, 8.6.1, 8.6.3.
AND

5. (i) Prove: for any integers $n_1, \ldots, n_k$, $\dim_{\mathbb{Q}} \mathbb{Q}(\sqrt{n_1}, \ldots, \sqrt{n_k}) \leq 2^k$.
   (ii) NOW suppose that all of the n_i are distinct primes. Prove that this dimension $= 2^k$. (Minimal polynomial and automorphisms.)

6. (i) True or False & Why: S_4 is generated by $(1, 2, 3, 4)$ together with any transposition (i.e., with any 2-cycle).
   (ii) True or False & Why: S_6 is generated by $(1, 2, 3, 4, 5, 6)$ together with any transposition.