HOMEWORK 9. DUE FRIDAY MARCH 14

HAND IN: 5.3.2, 5.3.5, 5.3.8, 5.4.1, 5.4.3

AND:

1. Let p be prime and let $G = GL_2(\mathbb{Z}_p)$ (from Homework 7-8).
   (i) Prove that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generates a Sylow p-subgroup of G.
   (ii) Prove that G has exactly $p+1$ Sylow p-subgroups.

2. (i) For each prime p dividing $|S_5|$ find a Sylow p-subgroup of S_5. How many Sylow p-subgroups are there for each p?
   (ii) Same for S_6 and S_7.

3. Let p be a prime.
   (i) How many elements of order p are there in S_p?
   (ii) How many Sylow p-subgroups are there in S_p?
   (iii) How many Sylow 5-subgroups are there in A_5?
   (iv) How many conjugacy classes of elements of order 5 are there in A_5? (Hint: not 1.)

DO NOT HAND IN: 5.3.1, 5.3.2, 5.4.2, 5.4.4, 5.4.9, 5.4.11

AND:

4. Let p be a prime. Consider the set G of all functions $f : \mathbb{Z}_p \to \mathbb{Z}_p$ defined by $f(x) = ax + b$ where $a, b \in \mathbb{Z}_p$ and $a \neq 0$.
   (i) Prove: G is a group. What is $|G|$?
   (ii) Describe a Sylow q-subgroup of G for each prime q dividing $|G|$. (The answers are very different for $q = p$ and $q \neq p$.)
   (iii) Exactly how many Sylow q-subgroups are there for each such q?

5. Prove: if P is a Sylow p–subgroup of G then $N_G(N_G(P)) = N_G(P)$.

6. T or F. If $|G|$ and n are relatively prime and if G acts on a set X having exactly n elements, then every element of G induces the identity permutation of X.