HOMEWORK 7-8. DUE FRIDAY MARCH 7

HAND IN: 5.1.3, 5.1.6, 5.1.10, 5.1.13, 5.1.17, 5.1.18, 5.2.3.

AND:

1. Prove that $\mathbb{Z}[\sqrt{-6}]$ is not a UFD by factoring some integer.

2. (i) Let K be a finite field and let $G = \{a \in K^\times \mid a = b^2 \text{ for some } b \in K\}$. Show that either $G = K^\times$ or G is a subgroup of index 2 in $K^\times$.

(ii) Prove: If p is prime then 2, 3 or 6 is a square in $\mathbb{Z}_p$.

(iii) Prove: $(x^2 - 2)(x^2 - 3)(x^2 - 6)$ has a root in $\mathbb{Z}_p$ for every prime p, but has no root in $\mathbb{Z}$.

3. Let p be prime. The *general linear group* $GL_2(\mathbb{Z}_p)$ is the group of all invertible 2×2 matrices with entries in $\mathbb{Z}_p$ (that is, $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0.$ in $\mathbb{Z}_p$). Prove that $|GL_2(\mathbb{Z}_p)| = p(p^2 - 1)(p - 1)$ (Hint: Use any a, b not both 0, and then any $(c, d) \in \mathbb{Z}_p^2$ not a multiple of (a, b)).

4. Prove: if a group G has a subgroup of index n, then G has a proper normal subgroup of index at most $n!$ (Example 5.1.6).

DO NOT HAND IN: 5.1.5, 5.1.7, 5.1.8, 5.1.9, 5.2.1.

AND

5. If A is a ring, let $Aut(A)$ denote the set of all automorphisms of A.

(i) If $\varphi\colon A \to B$ is an isomorphism of rings, prove that $\varphi^{-1}\colon S \to R$ is also an isomorphism.

(ii) Show that $Aut(A)$ is a group under composition of functions.

(iii) If $u \in A^\times$ show that $\varphi_u\colon a \to u^{-1}au$ (for $a \in A$) is in $Aut(A)$.

(iv) Show that $\{\varphi_u \mid u \in A^\times\}$ is a normal subgroup of $Aut(A)$.