HOMEWORK 4-5. DUE FRIDAY FEBRUARY 15

HAND IN: 6.4.2, 6.4.6, 6.4.7(a),(b), 6.4.12, 6.4.13, 6.4.15, 6.5.4, 6.5.6, 6.5.11, 6.5.12, 6.5.23.

AND:

1.

(i) Describe the group $\mathbb{Z}[i]^\times$ of invertible elements (i.e., units) in the ring $\mathbb{Z}[i]$ of Gaussian integers.

(ii) Let $\mathbb{H}_\mathbb{Z} = \{a \cdot 1 + b \cdot i + c \cdot j + d \cdot k \in \mathbb{H} : a, b, c, d \in \mathbb{Z}\}$ be the set of integer quaternions. Show that $\mathbb{H}_\mathbb{Z}$ is a subring of $\mathbb{H}$ and prove that $(\mathbb{H}_\mathbb{Z})^\times = Q$, the Quaternion group of 8 elements.

2. Let α be a (real or) complex root of the equation $x^3 + x + 3 = 0$. Let $K = \mathbb{Q}[\alpha] = \{a + b\alpha + c\alpha^2 | a, b, c \in \mathbb{Q}\}$. Show that:

(i) K is a subfield of $\mathbb{C}$; and

(ii) The three different choices for α produce isomorphic fields K.

3. (i) Prove: $Q(\mathbb{Z}[\sqrt{3}]) \cong \mathbb{Q}[\sqrt{3}]$.

(ii) Let D be an integral domain. Let S be any subset of D such that $0 \notin S$, $1 \in S$ and $a, b \Longrightarrow ab \in S$. Show that $\{d/s \mid d \in D, s \in S\}$ is the subring of $Q(D)$ generated by D and $\{1/s \mid s \in S\}$.

DO NOT HAND IN: 6.4.1, 6.4.3, 6.4.4, 6.4.5, 6.4.8, 6.4.10, 6.4.14, 6.5.1, 6.5.5, 6.5.15, 6.5.18.

AND

4.

(i) Let A be a ring with 1, and let a be an invertible element (i.e., unit) of A. Show that $x \to a^{-1}xa$ is an automorphism of A.

(ii) Prove that ring of quaternions $\mathbb{H}$ has INFINITELY MANY different subfields isomorphic to the complex numbers.

5. Let A be a commutative ring. Fix a polynomial $g \in A[x]$. For $f(x) \in A[x]$ let $f(g)$ denote the polynomial obtained by substituting g in place of x, and then "expanding" to a polynomial.

(i) Prove: $\varphi_g \colon f(X) \to f(g)$ is a homomorphism $A[X] \to A[X]$.

(ii) If A is a field, prove that φ_g is an automorphism if and only if $g = ax + b$ for some $a \in A^\times$, $b \in A$.