HOMEWORK 1. DUE FRIDAY JANUARY 18

HAND IN:
6.1.3, 6.1.4, 6.1.5, 6.1.9, 6.1.11, 6.1.13, 6.1.16, 6.1.17

AND:

1. Let $\binom{n}{k}$ denotes the binomial coefficient $n!/k!(n-k)!$ for $0 \leq k \leq n$ (and $0!$ is defined to be 1). Let A be a commutative ring with the identity element 1_A.

(i) Use INDUCTION to prove the Binomial Theorem in A: For all integers $n \geq 1$ and all $a, b \in A$,

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a b^{n-1} + b^n.$$

(ii) If $2 \cdot 1_A = 0$, show that $(a+b)^2 = a^2 + b^2$.

(iii) Generalize (ii) to the case in which $p \cdot 1_A = 0$ for a *prime* p.

2. (the division ring $\mathbb{H}$ of quaternions. Compare with the definition of the quaternion group Q). Consider the following matrices with complex coefficients.

$$i = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $k = ij$.

(i) Verify that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$, where 1 stands for the identity matrix; and $\{\pm 1, \pm i, \pm j, \pm k\}$ is just the quaternion group Q

(ii) Let $\mathbb{H} = \mathbb{R} \cdot 1 + \mathbb{R} \cdot i + \mathbb{R} \cdot j + \mathbb{R} \cdot k = \{a \cdot 1 + b \cdot i + c \cdot j + d \cdot k \mid a, b, c, d \in \mathbb{R}\}$. Prove that $\mathbb{H}$ is a noncommutative subring of the ring of all 2×2 matrices with complex entries.

(iii) Prove that $\mathbb{R} \cdot 1 + \mathbb{R} \cdot i$, $\mathbb{R} \cdot 1 + \mathbb{R} \cdot j$ and $\mathbb{R} \cdot 1 + \mathbb{R} \cdot k$ are subrings of $\mathbb{H}$, and *each* of these is isomorphic to $\mathbb{C}$.

(iv) Prove that *every* nonzero element of $\mathbb{H}$ has an inverse in $\mathbb{H}$ (Hint: $a1 - bi - cj - dk$ is almost what you need.)

(v) Prove that $\mathbb{H}$ is a 4-dimensional vector space over $\mathbb{R}$ ($\mathbb{H}$ is called the "division algebra" of *Hamilton's quaternions*, discovered more than 100 years ago.)

3. (A new look at polynomials.) Let A be a commutative ring with 1. Let S denote the set of all infinite sequences $(a_n)_{n=0}^{\infty} = (a_0, a_1, a_2, a_3, \cdots)$ such that at most finitely many of the a_n are not zero. (More precisely, there is an integer N such that $a_n = 0$ whenever $n \geq N$) Define addition and multiplication of elements of S by:

$$(a_n)_{n=0}^{\infty} + (b_n)_{n=0}^{\infty} = (a_n + b_n)_{n=0}^{\infty}$$

and

$$(a_n)_{n=0}^{\infty} (b_n)_{n=0}^{\infty} = (c_n)_{n=0}^{\infty} \ ,$$

where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$.

(i) Prove: S is a commutative ring with 0 element $(0, 0, 0, 0, \cdots)$ and identity element $(1, 0, 0, 0, \cdots)$. (Make sure S is closed under addition and multiplication before you start worrying about identities.)

(ii) Prove that $a \to (a, 0, 0, 0, 0, 0, \cdots)$ is an isomorphism from A onto a subring of S. *We'll identify A with its image under this isomorphism.*

(iii) Let $x = (0, 1, 0, 0, 0, 0, \cdots)$. Prove: every element of S can be written $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ for some $a_0, \ldots, a_n \in A$.

DO NOT HAND IN:

6.1.1, 6.1.2, 6.1.6, 6.1.8, 6.1.10, 6.1.12, 6.1.15.
AND

4. (i) Prove: $H = \{\sigma \in S_n \mid \sigma(n) = n\}$ is a subgroup of S_n isomorphic to S_{n-1}.

(ii) Since $|S_n|/|S_{n-1}| = n$, there are exactly n distinct left cosets of H in S_n,

Choose permutations $\sigma_1, \ldots, \sigma_n \in S_n$ such that $\sigma_1 H, \ldots, \sigma_n H$ are these n distinct left cosets. PROVE that your choice of n permutations is right.

5. Prove that there exists a subgroup H of S_n such that $H \cong S_k \times S_{n-k}$. Deduce that $\binom{n}{k} = |G/H|$ (in particular, this implies that $\binom{n}{k} \in \mathbb{N}$.)

6. (Klein four-group) Prove that the subset

$$K = \{e,\ (1,2)(3,4),\ (1,3)(2,4),\ (1,4)(2,3)\}$$

of S_4 is a normal subgroup of S_4. Also prove that $K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.