HOMEWORK 7. DUE MONDAY NOVEMBER 19

HAND IN:

#2.7.5, 2.7.6, 2.7.10, 2.7.11

AND

1. (a) Consider the group $\mathbb{Q}$ of rational numbers under addition. Prove that $\mathbb{Q}/\mathbb{Z}$ is an infinite group. Prove also that each element of the quotient group $\mathbb{Q}/\mathbb{Z}$ has finite order.

(b) Let $H = \{x \in \mathbb{C}^\times \mid x \text{ has finite order}\}$. Show that H is a subgroup of $\mathbb{C}^\times$, and that the quotient group $\mathbb{C}^\times/H$ has EXACTLY one element of finite order.

2. Prove that for any elements a, b in a finite group G, the elements ab and ba have the same order. (Hint: show that these elements are conjugate to each other)

3. (i) Prove that the set of all 3-cycles generates the alternating group A_n whenever $n \geq 3$. (Hint: $(1,2)(2,3) = (1,2,3)$.)

(ii) Use (i) to prove that A_5 is generated by the two elements $(1,2,3)$ and $(1,2,3,4,5)$.

DO NOT HAND IN:

#2.7.1, 2.7.2, 2.7.3, 2.7.12

AND

4. Prove that the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Deduce that there are infinitely many primes. (Hint: assume, by contradiction, that there are only l primes, hence each $n \in \mathbb{N}$ has a unique factorization $n = p_1^{a_1} p_2^{a_2} \cdots p_l^{a_l}$)

5. For $d \in \mathbb{N}$ and $b \in \mathbb{Z}$ let $A_{d,b} = d \cdot \mathbb{Z} + b = \{da + b \mid a \in \mathbb{Z}\}$ (that is, each $A_{d,b}$ is an arithmetic progression). We say that a subset S of $\mathbb{Z}$ is *open* if S is a union of (finitely or infinitely many) subsets $A_{d,b}$.

(i) Prove that for each prime p the subset $\mathbb{Z} \setminus A_{p,0}$ is open.

(ii) Let $A := \bigcup_p A_{p,0}$. Prove that $A = \mathbb{Z} \setminus \{-1, 1\}$.

(iii) Deduce that there are infinitely many primes (Hint: Show that if there are finitely many primes then $\mathbb{Z} \setminus A$ is open).

6. (i) Prove: If $\frac{a}{b} + \frac{c}{d} \in \mathbb{Z}$, where $a, b, c, d \in \mathbb{Z}$ and $(a, b) = 1 = (c, d)$, then $b = \pm d$.

(ii) Prove: $\forall n \geq 2$: $\displaystyle\sum_{i=1}^{n} \frac{1}{i} \notin \mathbb{Z}$. 2. Prove: If $a \in \mathbb{Z}$ is even then $(a^{2^m} + 1, a^{2^n} + 1) = 1$ for any distinct integers m, n. Deduce that there are infinitely many primes.